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Strongest relation, cosets and middle cosets of $AQFSC(G)$

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Abstract: The aim of this paper is to introduce strongest relation, cosets and middle cosets of anti Q-fuzzy subgroups of G with respect to t-conorm C. We investigate equivalent characterizations of them and we construct a new group induced by them, and give the homomorphism theorem between them.

Keywords: Fuzzy algebraic structures; Norms; Anti Q-fuzzy subgroups; Homomorphisms.

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1. Introduction and Preliminaries

In mathematics and abstract algebra, group theory studies the algebraic structures known as groups. The concept of a group is central to abstract algebra. Other well-known algebraic structures, such as rings, fields, and vector spaces, can all be seen as groups endowed with additional operations and axioms. Groups recur throughout mathematics, and the methods of group theory have influenced many parts of algebra. Linear algebraic groups and Lie groups are two branches of group theory that have experienced advances and have become subject areas in their own right.

The concept of a fuzzy set was introduced by Zadeh [1], and it is now a rigorous area of research with manifold applications ranging from engineering and computer science to medical diagnosis and social behavior studies. Conorms are operations which generalize the logical conjunction and logical disjunction to fuzzy logic. They are a natural interpretation of the conjunction and disjunction in the semantics of mathematical fuzzy logics and they are used to combine criteria in multi-criteria decision making. Yuan and Lee [2] defined the fuzzy subgroup and fuzzy subring based on the theory of falling shadows. Also Solairaju and Nagarajan [3] introduced the notion of Q- fuzzy groups. The author by using norms, investigated some properties of fuzzy algebraic structures [4–6].

In [7] the author introduced the notion of anti $Q-$fuzzy subgroups of G with respect to t-conorm C and study their important properties. In this work, we introduce strongest relation with respect anti Q-fuzzy subgroups of G with respect to t-conorm C and obtain some properties of them. Next, we define the middle coset of anti Q-fuzzy subgroups of G with respect to t-conorm C and investigate some results about them. Finally, we define new group under new operations of them and we prove isomorphism between them.

2. Preliminaries

In this section, we recall some of the fundamental concepts and definition, which are necessary for this paper. For more details we refer readers to [8–11].

Definition 1. A group is a non-empty set G on which there is a binary operation $(a, b) \to ab$ such that

1. if a and b belong to G then ab is also in G (closure),
2. $a(bc) = (ab)c$ for all $a, b, c \in G$ (associativity),
3. there is an element $e \in G$ such that $ae = ea = a$ for all $a \in G$ (identity),
4. if $a \in G$, then there is an element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$ (inverse).

One can easily check that this implies the unicity of the identity and of the inverse. A group G is called abelian if the binary operation is commutative, i.e., $ab = ba$ for all $a, b \in G$.

Remark 1. There are two standard notations for the binary group operation, either the additive notation, that is $(a, b) \to a + b$ in which case the identity is denoted by 0, or the multiplicative notation, that is $(a, b) \to ab$ for which the identity is denoted by e.

Definition 2. Let $(G,.),(H,.)$ be any two groups. The function $f : G \to H$ is called a homomorphism (anti-homomorphism) if $f(xy) = f(x)f(y)(f(xy) = f(y)f(x))$, for all $x,y \in G$.

Definition 3. Let G be an arbitrary group with a multiplicative binary operation and identity e. A fuzzy subset of G, we mean a function from G into $[0,1]$. The set of all fuzzy subsets of G is called the $[0,1]$-power set of G and is denoted $[0,1]^G$.

Definition 4. A t-conorm C is a function $C : [0,1] \times [0,1] \to [0,1]$ having the following four properties:

- (C1) $C(x,0) = x$,
- (C2) $C(x,y) \leq C(x,z)$ if $y \leq z$,
- (C3) $C(x,y) = C(y,x)$,
- (C4) $C(x,C(y,z)) = C(C(x,y),z)$, for all $x,y,z \in [0,1]$.

Example 1. Following are some examples of t-conorms:

1. Standard union t-conorm $C_m(x,y) = \max\{x,y\}$.
2. Bounded sum t-conorm $C_b(x,y) = \min\{1, x+y\}$.
3. Algebraic sum t-conorm $C_p(x,y) = x + y - xy$.
4. Drastic T-conorm
$$C_D(x,y) = \begin{cases} y & \text{if } x = 0 \\ x & \text{if } y = 0 \\ 1 & \text{otherwise,} \end{cases}$$
dual to the drastic T-norm.
5. Nilpotent maximum T-conorm, dual to the nilpotent minimum T-norm:
$$C_{nM}(x,y) = \begin{cases} \max\{x,y\} & \text{if } x+y < 1 \\ 1 & \text{otherwise.} \end{cases}$$
6. Einstein sum (compare the velocity-addition formula under special relativity) $C_{H_2}(x,y) = \dfrac{x+y}{1+xy}$ is a dual to one of the Hamacher t-norms. Note that all t-conorms are bounded by the maximum and the drastic t-conorm: $C_{\max}(x,y) \leq C(x,y) \leq C_D(x,y)$ for any t-conorm C and all $x,y \in [0,1]$.

Recall that t-conorm C is idempotent if for all $x \in [0,1]$, we have that $C(x,x) = x$.

Lemma 1. *Let C be a t-conorm, then*
$$C(C(x,y),C(w,z)) = C(C(x,w),C(y,z)),$$
for all $x,y,w,z \in [0,1]$.

Definition 5. Let $(G,.)$ be a group and Q be a non empty set. $\mu \in [0,1]^{G\times Q}$ is said to be an anti Q-fuzzy subgroup of G with respect to t-conorm C if the following conditions are satisfied:

1. $\mu(xy,q) \leq C(\mu(x,q),\mu(y,q))$,
2. $\mu(x^{-1},q) \leq \mu(x,q)$, for all $x,y \in G$ and $q \in Q$.

Throughout this paper the set of all anti Q-fuzzy subgroups of G with respect to t-conorm C will be denoted by $AQFSC(G)$.

Lemma 2. *Let $\mu \in AQFSC(G)$ and C be idempotent t-conorm, then $\mu(e_G,q) \leq \mu(x,q)$ for all $x \in G$ and $q \in Q$.*

Proposition 1. *Let C be idempotent t-conorm, then $\mu \in AQFSC(G)$ if and only if*
$$\mu(xy^{-1},q) \leq C(\mu(x,q),\mu(y,q))$$

for all $x, y \in G$ and $q \in Q$.

Definition 6. We say that $\mu \in AQFSC(G)$ is a normal if $\mu(xyx^{-1}, q) = \mu(y, q)$ for all $x, y \in G$ and $q \in Q$. We denote by $NAQFSC(G)$ the set of all normal anti Q-fuzzy subgroups of G with respect to t-conorm C.

Definition 7. Let $(G, .), (H, .)$ be any two groups such that $\mu \in AQFSC(G)$ and $\nu \in AQFSC(H)$. The product of μ and ν, denoted by $\mu \times \nu \in [0,1]^{(G\times H)\times Q}$, is defined as

$$(\mu \times \nu)((x, y), q) = C(\mu(x, q), \nu(y, q))$$

for all $x \in G, y \in H, q \in Q$.

Throughout this paper, H denotes an arbitrary group with identity element e_H.

Proposition 2. *If $\mu \in AQFSC(G)$ and $\nu \in AQFSC(H)$, then $\mu \times \nu \in AQFSC(G \times H)$.*

3. strongest relation, cosets and middle cosets of $QFST(G)$

Definition 8. Let $\mu \in [0,1]^{G\times Q}$ and $\nu \in [0,1]^{(G\times G)\times Q}$. We say that ν is the strongest relation of G with respect to μ if

$$\nu((x, y), q) = C(\mu(x, q), \mu(y, q)).$$

Proposition 3. *Let $\mu \in [0,1]^{G\times Q}$ and ν be the strongest relation of G with respect to μ. Then $\mu \in AQFSC(G)$ if and only if $\nu \in AQFSC(G \times G)$.*

Proof. Let $\mu \in AQFSC(G)$, then,

1. for all $(x_1, x_2), (y_1, y_2) \in G \times G$ and $q \in Q$ we have

$$\begin{aligned}\nu((x_1, x_2)(y_1, y_2), q) &= C(\mu(x_1, x_2), q), \mu(y_1, y_2), q)) \\ &\leq C(C(\mu(x_1, q), \mu(x_2, q)), C(\mu(y_1, q), \mu(y_2, q))) \\ &= C(C(\mu(x_1, q), \mu(y_1, q)), C(\mu(x_2, q), \mu(y_2, q))) \\ &= C(\nu((x_1, y_1), q), \nu((x_2, y_2), q))\end{aligned}$$

and then

$$\nu((x_1, x_2)(y_1, y_2), q) \leq C(\nu((x_1, y_1), q), \nu((x_2, y_2), q)).$$

2. For all $(x, y) \in G \times G$ and $q \in Q$ we obtain that

$$\begin{aligned}\nu((x, y)^{-1}, q) &= \nu((x^{-1}, y^{-1}), q) \\ &= C(\mu(x^{-1}, q), \mu(y^{-1}, q)) \\ &\leq C(\mu(x, q), \mu(y, q)) \\ &= \nu((x, y), q).\end{aligned}$$

Then $\nu \in AQFSC(G \times G)$.

Conversely, assume that $\nu \in AQFSC(G \times G)$.

1. Let $x_1, x_2, y_1, y_2 \in G$ with $x_2 = y_2 = e_G$ and $q \in Q$. From Lemma 2 we get that $\mu(e, q) \leq \mu(x_1 y_1, q)$ and so

$$\begin{aligned}\mu(x_1 y_1, q) &= C(\mu(x_1 y_1, q), \mu(e_G, q)) \\ &= C(\mu(x_1 y_1, q), \mu(x_2 y_2, q)) \\ &= \nu((x_1 y_1, x_2 y_2), q) \\ &= \nu((x_1, x_2)(y_1, y_2), q)\end{aligned}$$

$$\begin{aligned}
&\leq C(\nu((x_1,x_2),q),\nu((y_1,y_2),q))\\
&= C(C(\mu(x_1,q),\mu(x_2,q)),C(\mu(y_1,q),\mu(y_2,q)))\\
&= C(C(\mu(x_1,q),\mu(e,q)),C(\mu(y_1,q),\mu(e,q)))\\
&\leq C(C(\mu(x_1,q),\mu(x_1,q)),C(\mu(y_1,q),\mu(y_1,q)))\\
&= C(\mu(x_1,q),\mu(y_1,q)))
\end{aligned}$$

and thus

$$\mu(x_1y_1,q) \leq C(\mu(x_1,q),\mu(y_1,q)).$$

2. Let $x,y \in G$ with $y = e_G$ and $q \in Q$. Then by Lemma 2 we obtain

$$\begin{aligned}
\mu(x^{-1},q) &= C(\mu(x^{-1},q),\mu(e_G,q))\\
&\leq C(\mu(x^{-1},q),\mu(y^{-1},q))\\
&= \nu((x^{-1},y^{-1}),q)\\
&= \nu((x,y)^{-1},q) \leq \nu((x,y),q)\\
&= C(\mu(x,q),\mu(y,q))\\
&= C(\mu(x,q),\mu(e,q))\\
&\leq C(\mu(x,q),\mu(x,q))\\
&= \mu(x,q)
\end{aligned}$$

and then

$$\mu(x^{-1},q) \leq \mu(x,q).$$

Therefore $\mu \in AQFSC(G)$.

□

Definition 9. Let $\mu \in AQFSC(G)$, then middle coset $a\mu b : G \times Q \to [0,1]$ is defined by

$$(a\mu b)(x,q) = \mu(a^{-1}xb^{-1},q)$$

for all $x \in G, q \in Q$ and $a,b \in G$.

Proposition 4. *Let $\mu \in AQFSC(G)$, then $a\mu a^{-1} \in AQFSC(G)$ for any $a \in G$.*

Proof. Let $a,x,y \in G$ and $q \in Q$, then

1. $(a\mu a^{-1})(xy,q) = \mu(a^{-1}xya,q) = \mu(a^{-1}xaa^{-1}ya,q) \leq C(\mu(a^{-1}xa,q),\mu(a^{-1}ya,q))$
$= C((a\mu a^{-1})(x,q),(a\mu a^{-1})(y,q)),$
2. $(a\mu a^{-1})(x^{-1},q) = \mu(a^{-1}x^{-1}a,q) = \mu((a^{-1}xa)^{-1},q) \leq \mu(a^{-1}xa,q) = (a\mu a^{-1})(x,q).$

Thus $a\mu a^{-1} \in AQFSC(G)$ for any $a \in G$. □

Definition 10. Let $\mu \in AQFSC(G)$, then coset $a\mu : G \times Q \to [0,1]$ is defined by $(a\mu)(x,q) = \mu(a^{-1}x,q)$ for all $x \in G, q \in Q$ and $a \in G$.

Proposition 5. *Let $\mu \in AQFSC(G)$ and C be idempotent t-conorm, then $x\mu = y\mu$ if and only if $\mu(x^{-1}y,q) = \mu(y^{-1}x,q) = \mu(e_G,q)$ for all $x,y \in G$ and $q \in Q$.*

Proof. Let $x,y,g \in G$. If $x\mu = y\mu$, then $x\mu(x,q) = y\mu(x,q)$ and $\mu(x^{-1}x,q) = \mu(y^{-1}x,q)$ so $\mu(e_G,q) = \mu(y^{-1}x,q)$.

Also as $x\mu = y\mu$ so $x\mu(y,q) = y\mu(y,q)$ and $\mu(x^{-1}y,q) = \mu(y^{-1}y,q)$, which implies that $\mu(x^{-1}y,q) = \mu(e_G,q)$. Therefore we obtain that $\mu(x^{-1}y,q) = \mu(y^{-1}x,q) = \mu(e_G,q)$.

Conversely, let $\mu(x^{-1}y, q) = \mu(y^{-1}x, q) = \mu(e_G, q)$. Then

$$\begin{aligned}
x\mu(g,q) &= \mu(x^{-1}g, q) \\
&= \mu(x^{-1}yy^{-1}g, q) \\
&\leq C(\mu(x^{-1}y, q), \mu(y^{-1}g, q)) \\
&= C(\mu(e_G, q), \mu(y^{-1}g, q)) \\
&\leq C(\mu(y^{-1}g, q), \mu(y^{-1}g, q)) \\
&= \mu(y^{-1}g, q) = y\mu(g,q) \\
&= \mu(y^{-1}g, q) \\
&= \mu(y^{-1}xx^{-1}g, q) \\
&\leq C(\mu(y^{-1}x, q), \mu(x^{-1}g, q)) \\
&= C(\mu(e_G, q), \mu(x^{-1}g, q)) \\
&\leq C(\mu(x^{-1}g, q), \mu(x^{-1}g, q)) \\
&= \mu(x^{-1}g, q) = x\mu(g,q)
\end{aligned}$$

and $x\mu(g,q) = y\mu(g,q)$, which implies that $x\mu = y\mu$. □

Proposition 6. *Let $\mu \in AQFSC(G)$ and C be idempotent t-conorm. If $x\mu = y\mu$, then $\mu(x,q) = \mu(y,q)$ for all $x, y \in G$ and $q \in Q$.*

Proof. Since $x\mu = y\mu$, so from Proposition 5 we have $\mu(x^{-1}y, q) = \mu(y^{-1}x, q) = \mu(e_G, q)$ for all $x, y \in G$ and $q \in Q$. Now

$$\begin{aligned}
\mu(x,q) &= \mu(yy^{-1}x, q) \\
&\leq C(\mu(y,q), \mu(y^{-1}x, q)) \\
&= C(\mu(y,q), \mu(e_G, q)) \\
&\leq C(\mu(y,q), \mu(y,q)) \\
&= \mu(y,q) \\
&= \mu(xx^{-1}y, q) \\
&\leq C(\mu(x,q), \mu(x^{-1}y, q)) \\
&= C(\mu(x,q), \mu(e_G, q)) \\
&\leq C(\mu(x,q), \mu(x,q)) \\
&= \mu(x,q)
\end{aligned}$$

and thus $\mu(x,q) = \mu(y,q)$. □

Proposition 7. *If $\mu \in NAQFSC(G)$, then the set $\frac{G}{\mu} = \{x\mu : x \in G\}$ is a group with the operation $(x\mu)(y\mu) = (xy)\mu$.*

Proof.
1. If $x, y \in G$, then $xy \in G$. If $x\mu, y\mu \in \frac{G}{\mu}$ then $(x\mu)(y\mu) = (xy)\mu \in \frac{G}{\mu}$.
2. Let $x, y, z \in G$ then $x(yz) = (xy)z$. Now let $x\mu, y\mu, z\mu \in \frac{G}{\mu}$ so $(x\mu)[(y\mu)(z\mu)] = [(x\mu)](yz\mu) = (xyz)\mu = (xy)z\mu = (xy\mu)(z\mu) = [(xy\mu)](z\mu) = [(x\mu)(y\mu)](z\mu)$.
3. Let $x \in G$ then $xe_G = e_Gx = x$. Thus $(x\mu)(e_G\mu) = (xe_G\mu) = (e_Gx\mu) = x\mu$. If $x \in G$, then there is an element $x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = e_G$. Let $x\mu \in \frac{G}{\mu}$ there is an element $(x\mu)^{-1} = x^{-1}\mu \in \frac{G}{\mu}$ such that $(x\mu)(x\mu)^{-1} = (x\mu)(x^{-1}\mu) = (xx^{-1}\mu) = (x^{-1}x\mu) = e_G\mu = \mu$.

Hence $\frac{G}{\mu}$ is a group. □

Proposition 8. *Let $f : G \to H$ be a homomorphism of groups and let $\nu \in NAQFSC(H)$ and μ be homomorphic pre-image of ν. Then $\varphi : \frac{G}{\mu} \to \frac{H}{\nu}$ such that $\varphi(x\mu) = f(x)\nu$, for every $x \in G$, is an isomorphism of groups.*

Proof. Let $x, y \in G$ and $q \in Q$. Then

$$\varphi((x\mu)(y\mu)) = \varphi((xy)\mu) = f(xy)\nu = f(x)f(y)\nu = f(x)\nu f(y)\nu = \varphi(x\mu)\varphi(y\mu)$$

and so φ is a group homomorphism. Clearly φ is onto and we prove that φ is one-one. If $\varphi(x\mu) = \varphi(y\mu)$, then $f(x)\nu = f(y)\nu$ and from Proposition 5 we get

$$\nu(f(x)^{-1}f(y), q) = \nu(f(y)^{-1}f(x), q) = \nu(f(e_G), q),$$

so

$$\nu(f(x^{-1})f(y), q) = \nu(f(y^{-1})f(x), q) = \nu(f(e_G), q),$$

then

$$\nu(f(x^{-1}y), q) = \nu(f(y^{-1}x), q) = \nu(f(e_G), q),$$

which implies that

$$\mu(x^{-1}y, q) = \mu(y^{-1}x, q) = \mu(e_G, q),$$

and $x\mu = y\mu$ which implies that φ is one-one. Therefore φ will be an isomorphism of groups. □

Proposition 9. *Let $f : G \to H$ be an anti homomorphism of groups and let $\nu \in NQFST(H)$ and μ be anti homomorphic pre-image of ν. Then $\varphi : \frac{G}{\mu} \to \frac{H}{\nu}$ such that $\varphi(x\mu) = f(x)\nu$, for every $x \in G$, is an isomorphism of groups.*

Proof. Firstly, we prove that φ is an anti group homomorphism. Let $x, y \in G$ and $q \in Q$. Then

$$\varphi((x\mu)(y\mu)) = \varphi((xy)\mu) = f(xy)\nu = f(y)f(x)\nu = f(y)\nu f(x)\nu = \varphi(y\mu)\varphi(x\mu),$$

and so φ is an anti group homomorphism. Clearly φ is onto and we prove that φ is one-one. If $\varphi(x\mu) = \varphi(y\mu)$, then $f(x)\nu = f(y)\nu$ and from Proposition 5 we get

$$\nu(f(x)^{-1}f(y), q) = \nu(f(y)^{-1}f(x), q) = \nu(f(e_G), q),$$

so

$$\nu(f(x^{-1})f(y), q) = \nu(f(y^{-1})f(x), q) = \nu(f(e_G), q),$$

then

$$\nu(f(yx^{-1}), q) = \nu(f(xy^{-1}), q) = \nu(f(e_G), q),$$

which implies that

$$\mu(yx^{-1}, q) = \mu(xy^{-1}, q) = \mu(e_G, q)$$

and thus Proposition 5 gives us $x\mu = y\mu$ which implies that φ is one-one. Therefore φ will be an anti isomorphism of groups. □

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