



Some Applications of One and Two- Dimensional Fourier Series and Transforms

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

The paper provides a solid foundation in the fundamentals of one and two-dimensional Fourier series and transforms, examines the representation of periodic non-sinusoidal signals which can include current, voltage, voice, image etc, as sum of infinite trigonometrically series in sine and cosine terms, presents analysis of Fourier series with regard to some of its applications and modeling in electric circuits illustrated with corresponding numerical problems, also is being used the processing of images in its frequency domain rather than spatial domain associated with different filters, giving examples with algorithms developed in MATLAB and making visual comparisons for each method.

Keywords: Fourier series; Fourier transform; electric circuit; image processing.

1 Introduction

Conventional Fourier analysis has many schemes for different types of signals. They are Fourier transform (FT), Fourier series (FS), discrete-time Fourier transform (DTFT), and discrete Fourier transform (DFT).

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Using the Fourier analysis, the signal, which can be defined as a function (of time or space), whose values convey information of any mathematical or physical process that it presents, is decomposed in sinusoids, namely frequencies, whose amplitudes form so-called specter of frequencies of a signal [3].

Let us consider orthogonal set given as e^{inw_0t} , $n = 0, \pm 1, \pm 2, \dots$ in a segment $t \in \left[t_0, t_0 + \frac{2\pi}{w_0} \right]$. When a

signal is expanded in exponential series than we have to do with Fourier series of the function $f(t)$:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=-\infty}^{\infty} A_n e^{inw_0t}, \quad \text{where} \quad A_n = \frac{2\pi}{w_0} \int_{t_0}^{t_0 + \frac{2\pi}{w_0}} f(t) e^{-inw_0t} dt, \quad ,$$

$$A_0 = \frac{a_0}{2}, \quad A_n = \frac{a_n - ib_n}{2}, \quad A_{-n} = \frac{a_n + ib_n}{2}, \quad (n = 1, 2, 3, \dots) \quad \text{in order} \quad \int_{t_0}^{t_0 + \frac{2\pi}{w_0}} |f(t)| dt < \infty \quad \text{to be valid.}$$

Fourier transform is linear operator that given signal decomposes into its component frequencies and is defined as follows:

$$F(w) = \varphi[f](w) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt, \quad \text{where} \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) e^{iwt} dw, \quad F(w) = |F(w)| e^{i\theta(w)}$$

Inverse of Fourier transform is given by this formula $f(t) = \varphi^{-1}[\varphi[f](t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) e^{iwt} dw$ and using this the signal can be synthesized by adding all component frequencies.

Obviously that $F(w)$ plays the role of A_n in the first equation. Suppose that there are $2N + 1$ equidistant nodes $f_k = f(t_k)$, $\Delta = \frac{t_k}{k}$, $k = \overline{0, N-1}$, of discrete frequencies in a range from $-w_c$ to w_c . Consider frequencies $w_n = \frac{2\pi n}{\Delta N}$, for $N = -\frac{N}{2}, \dots, \frac{N}{2}$. We give approximation of Fourier transform in this range by the Riemann sum as follows:

$$F(w_n) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt \approx \sum_{k=0}^{N-1} f_k e^{-i w_n t} \Delta \quad \text{respectively,} \quad F(w) \approx \Delta \sum_{k=0}^{N-1} f_k e^{-ik \frac{2\pi n}{N}} = \Delta F_n, \quad \text{which gives}$$

the equation of discrete Fourier transform of the signal $f(t)$. Most efficient method for calculation of discrete Fourier transform, that separates it into sums of even terms and odd terms for $\frac{N}{2} - 1$ nodes and decreases time of calculations from $O(N^2)$ operations to $O(N \log_2 N)$ operations is algorithm called fast Fourier transform.

Let $W = e^{\frac{2\pi i}{N}}$ be sequence of complex numbers, than

$$F_n = \sum_{k=0}^{N-1} f_k W^{nk} = \sum_{k=0}^{\frac{N-2}{2}} f_{2k} e^{-\frac{2\pi i(2k)n}{N}} + \sum_{k=0}^{\frac{N-2}{2}} f_{2k+1} e^{-\frac{2\pi i(2k+1)n}{N}} = F_n^e + W^k F_n^o.$$

1.1 Some known results for Fourier transform

- i) $\varphi[f(t-a)](w) = e^{-iaw} \varphi[f](w)$ ii) $\varphi[f(at)](w) = \frac{1}{a} \varphi[f]\left(\frac{w}{a}\right)$
- iii) Convolution $h(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau) d\tau \Rightarrow \varphi[h](w) = \varphi[f](w) \cdot \varphi[g](w)$
- iv) $\varphi[f^{(n)}(t)](w) = (iw)^n \varphi[f](w)$ v) $\varphi^{-1}[f^{(n)}(w)](t) = (-it)^n \varphi^{-1}[f](t)$
- vi) $f(t) = \cos(2\pi w_0 t) \vee f(t) = \cos(2\pi w_0 t) \Rightarrow \varphi[f](w) = \delta(w-w_0) + \delta(w+w_0),$

where $\delta(t) = \begin{cases} \infty, t \neq 0 \\ 0, t = 0 \end{cases}, \int_{-\infty}^{\infty} \delta(t) dt = 1$

2 Application of Fourier Series and Transforms in Electric Circuits

For solving, analyzing and keeping in steady condition networks and electric circuits, which in practice sometimes are expressed by non-sinusoidal periodic functions, it suitable application of Fourier series, by analyzing phasors which represent complex numbers, respectively sinusoidal signals which show current or voltage values. Let us give the transformation of non-sinusoidal values to an infinite series [1,2,5].

Consider periodic function, with time period with T , $f(t-mT) = f(t), m \in N$ expanded in Fourier series as follows:

$$f(t) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kwt + b_k \sin kwt) = \frac{a_0}{2} + \sum_{k=1}^{\infty} A_k \cos(kwt + \alpha_k) = \frac{a_0}{2} + \sum_{k=1}^{\infty} B_k \sin(kwt + \beta_k),$$

where wt – frequencies and α_k, β_k – phases . By the known formulas $\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$ and $\cos(x \pm y) = \cos x \cos y \mp \sin y \sin x$ we obtain

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} A_k (\cos \alpha_k \cos kwt - \sin \alpha_k \sin kwt) = \frac{a_0}{2} + \sum_{k=1}^{\infty} B_k (\cos \beta_k \sin kwt + \sin \beta_k \cos kwt), \text{ so}$$

$$a_0 = \frac{2}{T} \int_0^T f(t) dt, a_k = A_k \cos \alpha_k = B_k \sin \beta_k = \frac{2}{T} \int_0^T f(t) \cos(kwt) dt \text{ and}$$

$$b_k = -A_k \sin \alpha_k = B_k \cos \beta_k = \frac{2}{T} \int_0^T f(t) \sin(kwt) dt,$$

$$|A_k| = |B_k| = \sqrt{a_k^2 + b_k^2}, \beta_k = \arctg \frac{a_k}{b_k} \text{ dhe } \alpha_k = -\arctg \frac{b_k}{a_k}$$

In [4] using the Fourier transform is analyzed square wave voltage signal, now applying Fourier series we will analyze unipolar voltage signal, which is given in the Fig. 1 below simulated by Vision Professional:

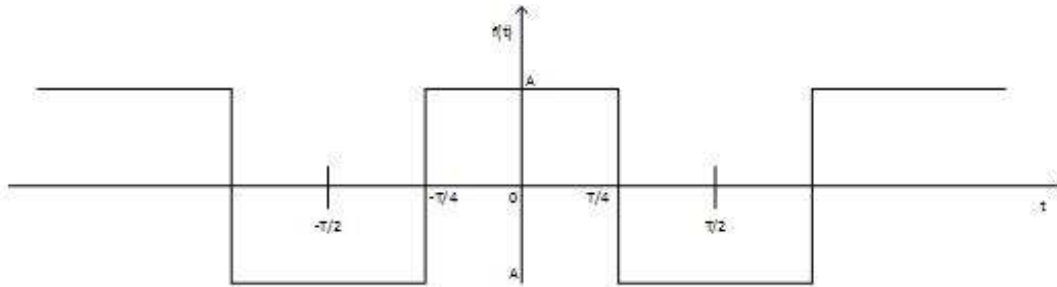


Fig. 1. Uni-polar voltage signal

From y-axis symmetry the signal is even that's why, $f(-t) = f(t)$ consequently it will not contain sine terms. We find Fourier coefficients in the exponential form:

$$A_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{T} \left[\int_0^{\frac{T}{4}} A dt + \int_{\frac{T}{4}}^{\frac{3T}{4}} -A dt + \int_{\frac{3T}{4}}^T A dt \right] = 0,$$

$$A_n = \frac{1}{T} \left[\int_0^{\frac{T}{4}} A e^{inwt} dt + \int_{\frac{T}{4}}^{\frac{3T}{4}} -A e^{inwt} dt + \int_{\frac{3T}{4}}^T A e^{inwt} dt \right] = \frac{A \left[2 \sin \frac{n\pi}{2} - \sin n\pi \right]}{n\pi}, \text{ respectively}$$

$$A_n = \begin{cases} 0, & n - \text{cift} \\ \frac{2A \sin \left(\frac{n\pi}{2} \right)}{n\pi}, & n - \text{tek} \end{cases}, \quad f(t) = \sum_{n=-\infty}^{\infty} A_n e^{inwt} = \frac{4A}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} \cos(2k+1)wt, \text{ which}$$

below is represented graphically using MATLAB for first three, four, five and seven terms where T=16:

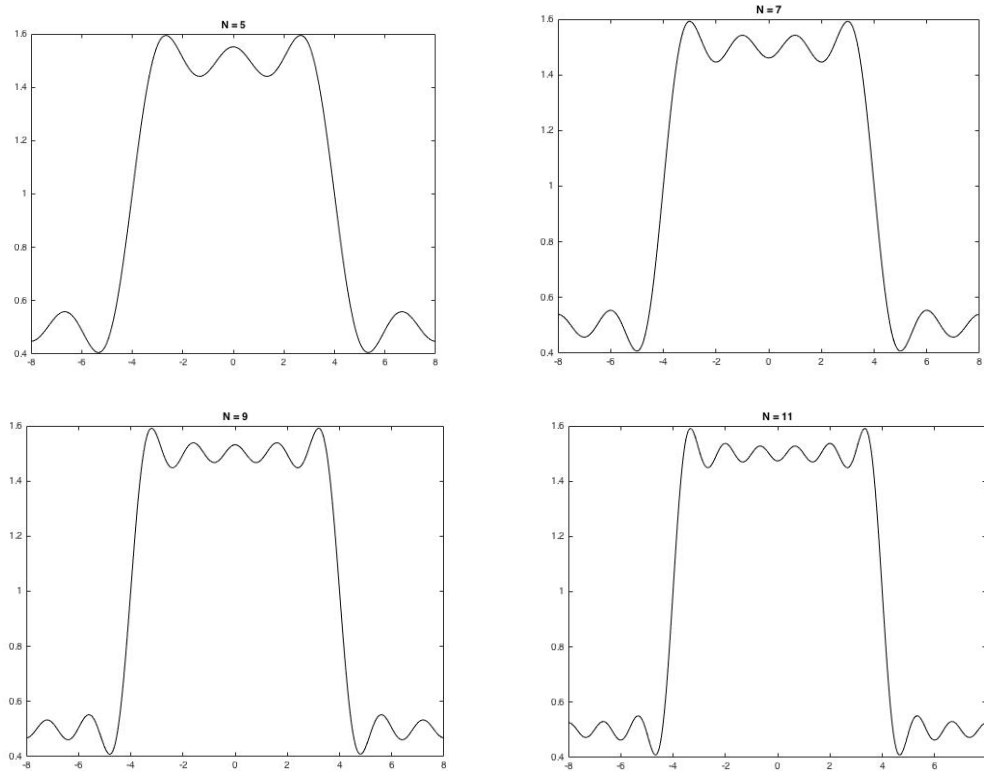


Fig. 2. Representation of unipolar voltage signal as cosine sums

where x-axis represents time and y-axis represents voltage.

This will be illustrated numerically for calculation of DC (direct current). Let $A=10V$. From above formulas can be concluded that $V(t) = V_{dc} + \sum_{k=1}^{\infty} V_k \cos(ktw - \alpha_{vk})$. DC term is 5 V. Amplitude of the primary

harmonic term is $\frac{4 \cdot 10}{\pi} = 12.7388$ and effective value (of the first term is)

$$A_1 = \frac{4 \cdot 10}{\pi\sqrt{2}} = \frac{40}{4.4406} = 9.007V$$

and other terms are displayed below in the Table 1:

Table 1. Primary terms of Fourier series

The Fourier series term	Voltage (V)
DC	+5V
First harmonic term	+9.007V
Third	-3.000V
Fifth	+1.800V
Seventh	-1.2860V
Ninth	+1,000V

While a signal represents a piece-wise continuous function $f : R \rightarrow C$, filter maps a given signal into a new signal, in order to modify it respectively to move the signal noise or represents an electrical impedance

which depends on the frequency of the signal current trying to pass. Next using the Fourier transform we will analyze passing the voltage step through high-pass filter.

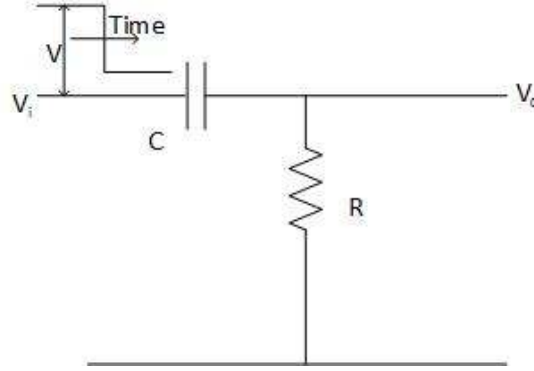


Fig. 3. A high-pass filter passing a voltage step

From Ohms law we have $V_0 = \frac{2\pi i v V_i}{\alpha + 2\pi i v}$, where $\alpha = \frac{1}{RC}$, R -resistance, C -circuit capacity. Input step

has amplitude V , as such by the function $H(x) = \int_{-\infty}^x \delta(t) dt$ where $\delta(t)$ is defined in the introduction

section, can be expressed as $\bar{V}_i(t) = VH(t)$. Its frequency will be $\frac{V}{2\pi i v} = V_i(v)$ and

$V_0(v) = \frac{V}{2\pi i v} \frac{2\pi i v}{\alpha + 2\pi i v} = \frac{V}{\alpha + 2\pi i v}$. The time variation of output voltage is given by the Fourier

transform $\bar{V}_0(t) = V \int_{-\infty}^{\infty} \frac{e^{2\pi i v t}}{\alpha + 2\pi i v} dv = \frac{V}{2\pi} \int_{-\infty}^{\infty} \frac{e^{itz}}{\alpha + iz} dz = -\frac{iV}{2\pi} \int_{-\infty}^{\infty} \frac{e^{itz}}{-\alpha i + z} dz$. Using the Cauchy

integral formula for integration of complex functions $2\pi i \operatorname{Re} s(f, a) = \int_C f(z) dz$ in our case the pole is

$z = \alpha i$ so $-\frac{iV}{2\pi} \int_C \frac{e^{itz}}{-\alpha i + z} dz = \frac{-2\pi i V e^{-t\alpha}}{2\pi} = V e^{-t\alpha}$ where contour zone consists i) real axis where

$dz = dx$, so is obtained the integral $\lim_{r \rightarrow \infty} -\frac{iV}{2\pi} \int_{-r}^r \frac{e^{ix}}{-\alpha i + x} dx$ and ii) semicircle with infinity radius where

the integrand vanishes. So $z = e^{i\varphi} = r(\cos \varphi + i \sin \varphi) \Rightarrow e^{itz} = e^{ir(\cos \varphi + i \sin \varphi)t}$ with real part $e^{-rt \sin \varphi}$,

which for $t > 0$ and upper part of the circle vanishes as radius tends to infinity, namely the time variation of output voltage is $V_0(t) = V e^{-\alpha t}$. Next we will give a numerical example.

In the Fig. 4, the circuit has source $V_s(t)$ of non-sinusoidal form, whose Fourier series is

$V_s(t) = 0.5 + \sum_{k=1}^{\infty} \frac{2 \sin((2k-1)\pi t)}{(2k-1)\pi}$. We will find voltage $V_0(t)$ in inductor and find the amplitudes of the corresponding spectrum.

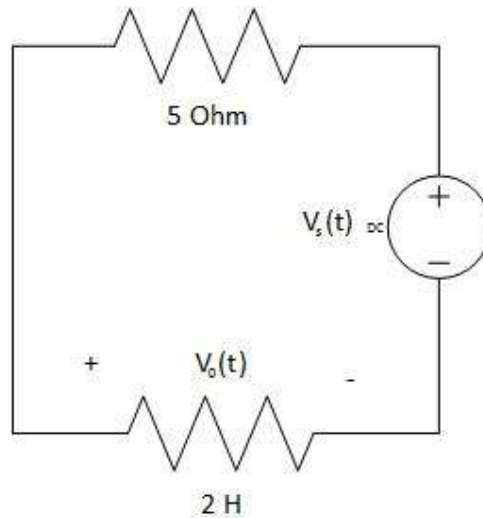


Fig. 4. Electric circuit

Output voltage will be $V_0(t) = \frac{2w_n i V_s(t)}{2n\pi i + 5}$, $w_n = n\pi$. DC component will be 0 for $w_n = 0$. Considering that sine part of phasor of AC (alternating current) is $\frac{2}{n\pi} \angle -90^\circ$ we obtain that

$$V_0(t) = \frac{2n\pi}{\sqrt{25 + 4n^2\pi^2}} \frac{2}{n\pi} \left(-\arctg \frac{2n\pi}{5} \right) = 4 \sum_{n=1}^{\infty} V_n \frac{\cos \left(n\pi t - \arctg \left(\frac{2(2n-1)\pi}{5} \right) \right)}{\sqrt{25 + 4(2n-1)^2\pi^2}} \quad \text{By}$$

calculations for four first terms is obtained this results:

Table 2. Primary terms of Fourier series

w	π	3π	5π	7π
V_0	0.5	0.2	0.13	0.1

3 Some Applications of Two-dimensional Fourier Transform in Image Processing

For a matrix of the dimension $M \times N$ two-dimensional discrete Fourier transform (DFT) and its inverse (IDFT), respectively, can be written as

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-\frac{2\pi i(Mvy+Nux)}{MN}} \quad \text{and} \quad f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{\frac{2\pi i(Mvy+Nux)}{MN}}$$

[6,7].

Next we represent some two-dimensional sinusoids:

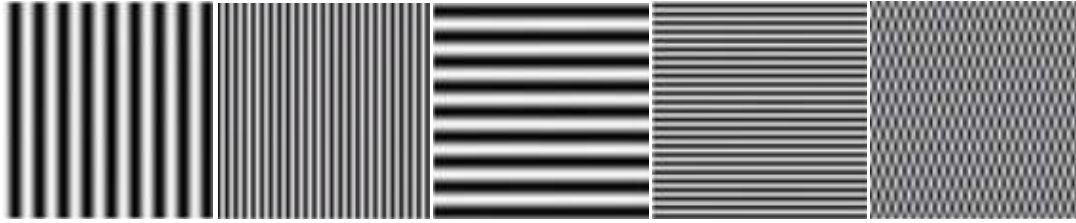


Fig. 5. Some two-dimensional sinusoids

Are displayed: first image $\left\{ \cos \frac{15\pi}{256} y, u = 0, v = \frac{15\pi}{256} \right\}$, second image $\left\{ \cos \frac{55\pi}{256} y, u = 0, v = \frac{55\pi}{256} \right\}$, third image $\left\{ \cos \frac{15\pi}{256} x, u = \frac{15\pi}{256}, v = 0 \right\}$, fourth image $\left\{ \cos \frac{55\pi}{256} x, u = \frac{55\pi}{256}, v = 0 \right\}$ and fifth image $\left\{ \cos \left(\frac{55\pi}{256} y \right) \cos \left(\frac{15\pi}{256} y \right), u = \frac{15\pi}{256}, v = \frac{55\pi}{256} \right\}$.

Idea is that the given image $f(x, y)$ with dimensions $M \times N$ is represented in domain frequencies $F(u, v)$, which by using the inverse Fourier transform returns to the given image, namely is decomposed into sum of two-dimensional weighted orthogonal functions in the same way as decomposition of vector into basis using scalar product, as follows:

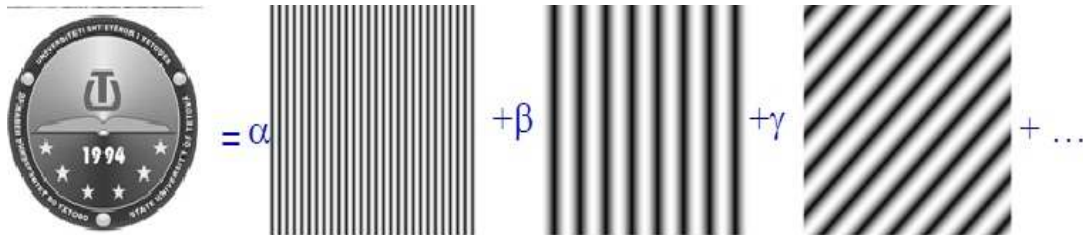


Fig. 6. Decomposition of an image as sums of sines and cosines

The value $F(u, v)_{u,v=0}$ is called dc-component and is for $M \times N$ times greater than average value of $f(x, y)$. In this case in visual way we analyze Fourier transform by calculation the magnitude or spectrum $F(u, v) (|F(u, v)|)$, which really is displayed as an image. By the following theorem is given the relation between spatial domain and frequency domain, at the same time the basic steps of DFT filtering in frequency domain.

Theorem 1. $g(x, y) = h(x, y) * f(x, y) \Leftrightarrow G(x, y) = H(x, y) F(x, y)$ and conversely $G(x, y) = H(x, y) * F(x, y) \Leftrightarrow g(x, y) = h(x, y) f(x, y)$ where $h(x, y) = IDFT(H(u, v)) \wedge H(x, y) = DFT(h(u, v))$.

From this theorem can be concluded that that the multiplication of two Fourier transforms corresponds to convolution of two functions which are their inverse of transforms in the spatial domain. So using the Fourier transform helps the acceleration of spatial filters.

We apply the Sobel filter, which uses kernels of dimension 3x3 convoluted with original image in order to calculate the derivative approximations of both axes vertical and horizontal. Kernel is given by the multiplication of matrices $S = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$. Let Abe taken as original image than G_x and G_y are images that contain derivative approximation points of horizontal and vertical axis respectively and are given by the formulas $G_x = S * A$ and $G_y = S^T * A$. For this application is taken the image for logo of Tetovo University, and are obtained these results:



Fig. 7. Sobel filter application

Are displayed: Original image, the image filtered in spatial and frequency domain, the image obtained by absolute value which correct magnitude when used complex numbers and threshold into a binary image, respectively.

Smoothing filters create a blurred image, attenuate the high frequencies and do not change the low frequencies of the Fourier transform. With $D(u, v)$ is denoted the distance between (u, v) and the center of the filter.

Let $a \geq 0$ be a fixed number. For the same image we use a smoothing filter defined by the formula

$$H(u, v) = e^{\left(\frac{-D^2(u, v)}{2D_0^2}\right)}$$

and are obtained these results:

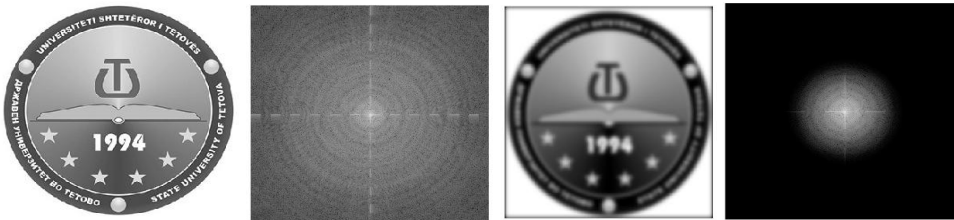


Fig. 8. Smoothing filter application

Are displayed: original image, Fourier spectrum of the image, the image with used filter and the spectrum of the image where is used the filter, respectively.

Sharpening filters show the edges of an image, attenuate the low frequencies and do not change high frequencies of the Fourier transform. The relation between sharpening filters denoted as $H_{sh}(u, v)$ and

smoothing filters denoted by $H_s(u, v)$, is given by this formula $H_{sh}(u, v) + H_s(u, v) = 1$. Let $a \geq 0$ be a fixed number. For the same image we use the sharpening filter given by formula $H(u, v) = 1 - e^{\left(\frac{-D^2(u, v)}{2D_0^2}\right)}$ and obtained these results:

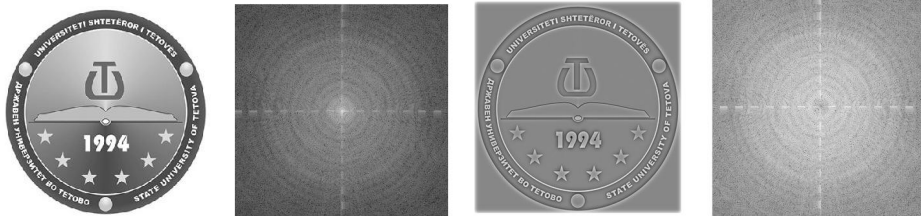


Fig. 9. Sharpening filter application

Are displayed: original image, Fourier spectrum of the image, the image with used filter and the spectrum of the image where is used the filter, respectively.

Band-stop filters are used to remove repetitive magnitude noise from an image, are like a narrow sharpen filter, but they novelize out frequencies other than the dc component, attenuate a selected frequency and do not change other frequencies of the Fourier transform. We apply the band-stop filter

$H(u, v) = \frac{(D(u, v))^{2n}}{(D_0)^{2n} + (D(u, v))^{2n}}$ to a noisy image (a ball) and are obtained these results:

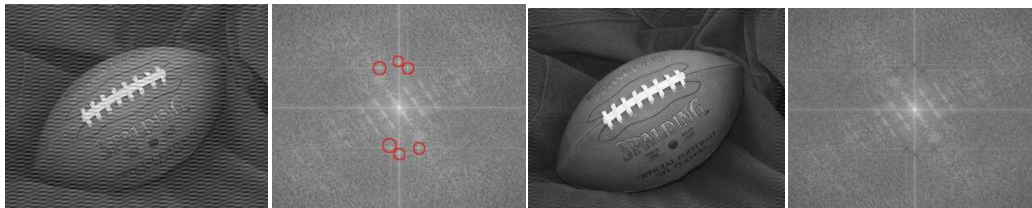


Fig. 10. Band-stop filter application

Are displayed: noisy image, Fourier spectrum of the noisy image (with selected filters that are attenuated), the image with used filter and the spectrum of the image where is used the filter, respectively.

4 Conclusions

In the first section of the paper are given general facts of Fourier series and transform as pre-requisites for developing a foundation in understanding the relationship between time domain-frequencies and communication signals. In the second section is given application of one-dimensional Fourier series and transform in electrical networks, by using conventional complex/vectors methods, as analyzing unipolar signal and passing the voltage step through high-pass filter illustrated with numerical examples. In the third section is being used the processing of images in its frequency domain rather than spatial domain associated with different filters, giving examples with algorithms developed in MATLAB and making visual description and comparison for each method.

Competing Interests

Authors have declared that no competing interests exist.

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