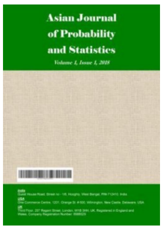

Asian Journal of Probability and Statistics

1(3): 1-8, 2018; Article no.AJPAS.41536

Refinements of Gaussian Tail Inequality

N. A. Rather[1] and T. A. Rather[1*]

[1] *Department of Mathematics, University of Kashmir, Srinagar, 190006, India.*

Authors' contributions

This work was carried out in collaboration between both authors. Author NAR designed the study,performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. Author TAR managed the analyses of the study. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/AJPAS/2018/v1i324541

Editor(s):

(1) Dr. Oguntunde, Pelumi Emmanuel, Department of Mathematics, College of Science and Technology, Covenant University, Ota, Ogun State, Nigeria.

Reviewers:

(1) Thomas L. Toulias, Technological Educational Institute of Athens, Greece.

(2) Yousri Slaoui, Laboratoire de Mathématiques et Applications UMR, France.

(3) Djaballah Khedidja, University of Science and Technology Houari Boumediene, Algeria.

Complete Peer review History: http://www.sciencedomain.org/review-history/26082

Received: ***2nd June 2018***

Accepted: ***21st August 2018***

Published: ***4th September 2018***

Original Research Article

ABSTRACT

In this paper, we first prove a theorem which gives considerably better bound for $0 \le t \le 1/2$ than Gaussian tail inequality (or tail bound for normal density) and thus is a refinement of Gaussian tail inequality in this case. Next we present an interesting result which provides a refinement of Gaussian tail inequality for $t > \sqrt{3}$. Besides, we also prove an improvement of Gaussian tail inequality for $0 < t \le 1/2$. Finally, we present a more general result which includes a variety of interesting results as special cases.

Keywords: Probability inequalities; Gaussian tail inequality; probability density function.

*Corresponding author: tariqrather@rediffmail.com

1 INTRODUCTION

The extremes of Gaussian random fields have wide applications in finance, spatial analysis, physical oceanography, and many other disciplines [1,2]. Tail probabilities of the extremes have been extensively studied in the literature, with its focus mostly on the development of approximations and bounds for the suprema [3,4,5,6,7,8]. Tail probabilities of other convex functions of Gaussian random process have also been studied. For instance, Liu [9] and Liu and Xu [10] derived the asymptotic approximations of the tail probabilities of the exponential integrals of Gaussian random fields; see also Liu and Xu [11]. Most of the sharp theoretical approximations developed in the literature hold only for constant variance fields, which also need certain smoothness conditions of the Gaussian random fields [8,12]. For the case of less smooth fields, the approximations involve the unknown Pickands constants [5]. Numerical methods for rare-event analysis of the suprema are studied in [13] and more thoroughly in [12]; see also Azas and Wschebor [14,15]. Simulation study for the exponential integrals of the Gaussian random fields has been studied in Liu and Xu [16,11,17].

2 REFINEMENT OF GAUSSIAN TAIL INEQUALITY

Let X be a random variable having normal distribution with mean 0 and variance 1. The following result known as Gaussian Tail inequality (GTI) or the tail bound for normal density (for reference see, Gordon R.D [18], Fan. P [19].

Theorem A. If $X \sim N(0,1)$, then for $t > 0$,

$$P\left(|X| > t\right) \leq \sqrt{\frac{2}{\pi}}\left(\frac{1}{t}\right)e^{-\frac{t^2}{2}}.$$

Here we first prove the following result which gives considerably better bound for $0 \leq t \leq \frac{1}{2}$ than the Gaussian Tail inequality (GTI)and thus is a refinement of the Gaussian Tail Inequality (GTI) in this case.

Theorem 1. If $X \sim N(0,1)$, then for $t > 0$,

$$P\left(|X| > t\right) \leq 1 - \sqrt{\frac{2}{\pi}}\,(t)e^{-\frac{t^2}{2}}.$$

Proof of Theorem 1. The probability density function (p.d.f) of X is given by

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty. \tag{2.1}$$

We then have,

$$P\left(X > t\right) = \int_0^\infty \phi(x)dx - \int_0^t \phi(x)dx. \tag{2.2}$$

Since $\phi(x)$ is an even function and a density of probability, we have,

$$\int_0^\infty \phi(x)dx = \frac{1}{2},$$

Hence from (2.2),

$$P\left(X > t\right) \leq \frac{1}{2} - \int_0^t \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{1}{e^{\frac{x^2}{2}}}\right)dx. \tag{2.3}$$

Since $x \leq t$, it holds

$$\frac{x^2}{2} \leq \frac{t^2}{2},$$

which implies,

$$e^{\frac{x^2}{2}} \leq e^{\frac{t^2}{2}}.$$

This gives,

$$\int_0^t \frac{1}{e^{\frac{x^2}{2}}}dx \geq \int_0^t \frac{1}{e^{\frac{t^2}{2}}}dx$$
$$= \frac{1}{e^{\frac{t^2}{2}}}\int_0^t dx = \frac{t}{e^{\frac{t^2}{2}}}.$$

Using this in (2.3), we get

$$P\left(X > t\right) \leq \frac{1}{2} - \int_0^t \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{t}{e^{\frac{t^2}{2}}}\right).$$

Hence by symmetry,

$$\begin{aligned} P\Big(|X|>t\Big) &= 2P\Big(X>t\Big) \\ &\le 2\left[\frac{1}{2}-\frac{t}{\sqrt{2\pi}\ e^{\ t^2/2}}\right] \\ &= 1-\sqrt{\frac{2}{\pi}}\ t\ e^{-\frac{t^2}{2}}. \end{aligned}$$

This completes the proof of Theorem 1.

Remark 1. To verify that the Theorem 1 gives a refinement of Gaussian Tail inequality (GTI) for $0<t\le 1/2$, we shall show that

$$1-\sqrt{\frac{2}{\pi}}\left(t\ e^{-\frac{t^2}{2}}\right)<\sqrt{\frac{2}{\pi}}\left(\frac{e^{-\frac{t^2}{2}}}{t}\right). \qquad (2.4)$$

Since, $e^{\frac{t^2}{2}}\le e^{\frac{1}{8}}$ for $0<t\le 1/2$, and it can be easily verified that $e^{\frac{1}{8}}<6/5$, so that $e^{\frac{t^2}{2}}<6/5$.

This implies,

$$e^{-\frac{t^2}{2}}>5/6, \text{ for } 0<t\le 1/2.$$

Now,

$$\begin{aligned} \sqrt{\frac{2}{\pi}}\left(te^{-\frac{t^2}{2}}\right)+\sqrt{\frac{2}{\pi}}\left(\frac{e^{-\frac{t^2}{2}}}{t}\right) &= \sqrt{\frac{2}{\pi}}\left(e^{-\frac{t^2}{2}}\right)\left(t+\frac{1}{t}\right) \\ &> \sqrt{\frac{1}{2}}\left(\frac{5}{6}\right)\Big(2\Big) \\ &= \frac{5}{3\sqrt{2}}>1. \end{aligned}$$

Hence,

$$\sqrt{\frac{2}{\pi}}\left(\frac{e^{-\frac{t^2}{2}}}{t}\right)>1-\sqrt{\frac{2}{\pi}}\left(t\ e^{-\frac{t^2}{2}}\right),$$

or

$$1-\sqrt{\frac{2}{\pi}}\left(t\ e^{-\frac{t^2}{2}}\right)<\sqrt{\frac{2}{\pi}}\left(\frac{e^{-\frac{t^2}{2}}}{t}\right),$$

which proves (2.4) for $0<t\le 1/2$.

Next we prove the following interesting result which provides a refinement of Gaussian Tail inequality (GTI) for $t>\sqrt{3}$.

Theorem 2. If $X\sim N(0,1)$, then for $t>0$,

$$P\Big(|X|>t\Big)\le\sqrt{\frac{2}{\pi}}\ e^{-\frac{t^2}{2}}\left(\frac{t^4-t^2+3}{t^5}\right).$$

Proof of Theorem 2. The p.d.f of X is given as in (2.1), which on differentiation gives,

$$\begin{aligned} \phi'(x) &= \frac{1}{\sqrt{2\pi}}\ e^{\frac{x^2}{2}}(-x) \\ &= -x\left(\frac{1}{\sqrt{2\pi}}\ e^{\frac{x^2}{2}}\right) \\ &= -x\phi(x). \end{aligned} \qquad (2.5)$$

Now

$$\begin{aligned} P(X>t) &= \int_t^\infty \phi(x)dx \\ &= \int_t^\infty \frac{x\phi(x)}{x}dx \\ &= -\int_t^\infty \frac{\phi'(x)}{x}dx. \qquad \text{(by (2.5))}. \end{aligned}$$

Integrating by parts, keeping $\phi'(x)$ for integration and $1/x$ for differentiation, we get

$$\begin{aligned} P(X>t) &= -\left[\frac{1}{x}\phi(x)\right]_t^\infty+\int_t^\infty \frac{-\phi(x)}{x^2}dx \\ &= \frac{\phi(t)}{t}-\int_t^\infty \frac{\phi(x)}{x^2}dx. \end{aligned} \qquad (2.6)$$

Now

$$\begin{aligned} \int_t^\infty \frac{\phi(x)}{x^2}dx &= \int_t^\infty \frac{x\phi(x)}{x^3}dx \\ &= -\int_t^\infty \frac{\phi'(x)}{x^3}dx \qquad \text{(by(2.5))} \\ &= -\int_t^\infty \left(\frac{1}{x^3}\right)\phi'(x)dx. \end{aligned}$$

Integrating by parts as before, we get

$$\int_t^\infty \frac{\phi(x)}{x^2}dx = -\left[\left(\frac{1}{x^3}\right)\phi(x)\right]_t^\infty + \int_t^\infty \phi(x)\left(\frac{-3}{x^4}\right)dx$$
$$= \frac{\phi(t)}{t^3} - 3\int_t^\infty \frac{\phi(x)}{x^4}dx.$$

Thus from (2.6),we have

$$P(X>t) = \frac{\phi(t)}{t} - \frac{\phi(t)}{t^3} + 3\int_t^\infty \frac{\phi(x)}{x^4}dx. \tag{2.7}$$

Again,

$$\int_t^\infty \frac{\phi(x)}{x^4}dx = \int_t^\infty \frac{x\,\phi(x)}{x^5}dx$$
$$= -\int_t^\infty \frac{\phi'(x)}{x^5}dx, \quad \text{(by(2.5))}$$
$$= -\left[\frac{\phi(x)}{x^5}\right]_t^\infty + \int_t^\infty \phi(x)\left(\frac{-5}{x^6}\right)dx$$
$$= \frac{\phi(t)}{t^5} - 5\int_t^\infty \frac{\phi(x)}{x^6}dx.$$

Hence from (2.7), we have

$$P(X>t) = \frac{\phi(t)}{t} - \frac{\phi(t)}{t^3} + 3\frac{\phi(t)}{t^5} - 15\int_t^\infty \frac{\phi(x)}{x^6}dx$$
$$= \phi(t)\left(\frac{1}{t} - \frac{1}{t^3} + \frac{3}{t^5}\right) - 15\int_t^\infty \frac{\phi(x)}{x^6}dx. \tag{2.8}$$

Since $\int_t^\infty \frac{\phi(x)}{x^6}dx \geq 0$, therefore, we have from (2.8),

$$P(X>t) \leq \phi(t)\left(\frac{1}{t} - \frac{1}{t^3} + \frac{3}{t^5}\right)$$
$$= \frac{\phi(t)}{t^5}\left(t^4 - t^2 + 3\right).$$

By symmetry,

$$P\left(|X|>t\right) = 2P(X>t)$$
$$\leq 2\,\frac{\phi(t)}{t^5}\left(t^4 - t^2 + 3\right)$$
$$= \sqrt{\frac{2}{\pi}}\,e^{-\frac{t^2}{2}}\left(\frac{t^4 - t^2 + 3}{t^5}\right).$$

This completes the proof of Theorem 2.

Remark 2. To verify that the Theorem 2 is a refinement of Gaussian Tail inequality (GTI) for $t>\sqrt{3}$, we note that $t>\sqrt{3}$ implies $t^2>3$, so that, $3-t^2<0$, which gives,

$$\left(\frac{t^4 - t^2 + 3}{t^5}\right) < \frac{t^4}{t^5} = \frac{1}{t}.$$

This implies,

$$\sqrt{\frac{2}{\pi}}\,e^{-\frac{t^2}{2}}\left(\frac{t^4 - t^2 + 3}{t^5}\right) < \sqrt{\frac{2}{\pi}}\,\frac{e^{-\frac{t^2}{2}}}{t},$$

which shows that the bound obtained in Theorem 2 is better than the bound given by Gaussian Tail inequality (GTI).

Our next Theorem gives an improvement of Theorem 1 and therefore Gaussian Tail inequality (GTI) for $0<t\leq 1/2$.

Theorem 3. If $X \sim N(0,1)$, then for $t>0$,

$$P\left(|X|>t\right) \leq 1 - \sqrt{\frac{2}{\pi}}\,e^{-\frac{t^2}{2}}\left(t + \frac{t^3}{3}\right).$$

Proof of Theorem 3. The p.d.f of X is given as in (2.1), while we have,

$$P(X>t) = \int_t^\infty \phi(x)dx$$
$$= \int_0^\infty \phi(x)dx - \int_0^t \phi(x)dx. \tag{2.9}$$

Integrating the 2nd integral in (2.9) by parts and using the fact that

$\phi'(x) = -x\phi(x)$, we get

$$\int_0^t \phi(x)dx = \int_0^t 1.\phi(x)dx$$

$$= \Big[x\ \phi(x)\Big]_0^t - \int_0^t x\ \phi'(x)dx$$

$$= t\phi(t) - \int_0^t -x^2\ \phi(x)dx$$

$$= t\phi(t) + \int_0^t x^2\phi(x)dx. \tag{2.10}$$

Since $\phi(x)$ is a decreasing function $x \geq 0$, it follows that $\phi(x) \geq \phi(t)$ for $x \leq t$.
Hence from (2.10), we get

$$\int_0^t \phi(x)dx \geq t\phi(t) + \phi(t)\int_0^t x^2 dx$$

$$= t\phi(t) + \phi(t)\left[\frac{x^3}{3}\right]_0^t$$

$$= t\phi(t) + \phi(t)\left[\frac{t^3}{3}\right]$$

$$= \phi(t)\left[t + \frac{t^3}{3}\right]. \tag{2.11}$$

Using (2.11) in (2.9) and the fact that

$\int_0^\infty \phi(x)dx = P(X > t) = \frac{1}{2}$, we get

$$P(X > t) \leq \frac{1}{2} - \phi(t)\left[t + \frac{t^3}{3}\right].$$

By symmetry, this implies

$$P\Big(|X| > t\Big) = 2P(X > t)$$

$$\leq 1 - 2\left(\frac{1}{\sqrt{2\pi}}\ e^{-\frac{t^2}{2}}\right)\left(t + \frac{t^3}{3}\right)$$

$$= 1 - \sqrt{\frac{2}{\pi}}\ e^{-\frac{t^2}{2}}\left(t + \frac{t^3}{3}\right).$$

This completes the proof of Theorem 3.

Remark 3. From (2.11), we have

$$\left(\frac{1}{\sqrt{2\pi}}\ e^{-\frac{t^2}{2}}\right)\left(t + \frac{t^3}{3}\right) \leq \int_0^t \phi(x)dx$$

$$\leq \int_0^\infty \phi(x)dx = \frac{1}{2},$$

so that,

$$\sqrt{\frac{2}{\pi}}\ e^{-\frac{t^2}{2}}\left(t + \frac{t^3}{3}\right) \leq 1.$$

Finally, in this section we present the following more general result, which include a variety of interesting results as special cases.

Theorem 4. If $X \sim N(0,1)$, then for $t > 0$,

$$P\Big(|X| > t\Big) = \sqrt{\frac{2}{\pi}}\ e^{-\frac{t^2}{2}}\left(\frac{1}{t} - \frac{1}{t^3} + \frac{1\cdot 3}{t^5} - \frac{1\cdot 3\cdot 5}{t^7} + \frac{1\cdot 3\cdot 5\cdot 7}{t^9} - \cdots\right).$$

For the proof of this theorem, we need the following lemma.

Lemma 1. If $\phi(x)$ is the p.d.f of the random variable $X \sim N(0,1)$, then for $t > 0$,

$$\int_t^\infty \frac{\phi(x)}{x^{2n}} = \frac{\phi(t)}{t^{2n+1}} - (2n+1)\int_t^\infty \frac{\phi(x)}{x^{2n+2}}dx, \text{ for } n = 0, 1, 2, ...$$

Proof of Lemma 1. The p.d.f of X is given as in (2.1).Integrating by parts, keeping $1/x^{2n+1}$ for differentiation and $\phi'(x)$ for integration and using the fact that $\phi'(x) = -x\phi(x)$, we get for any integer $n \geq 0$, and $t > 0$,

$$\int_t^\infty \frac{\phi(x)}{x^{2n}} = -\int_t^\infty \frac{\phi'(x)}{x^{2n+1}}dx$$

$$= -\left[\frac{1}{x^{2n+1}}\phi(x)\right]_t^\infty + \int_t^\infty -\frac{(2n+1)\phi(x)}{x^{2n+2}}dx$$

$$= \frac{\phi(t)}{t^{2n+1}} - \int_t^\infty (2n+1)\frac{\phi(x)}{x^{2n+2}}dx$$

$$= \frac{\phi(t)}{t^{2n+1}} - (2n+1)\int_t^\infty \frac{\phi(x)}{x^{2n+2}}dx.$$

This proves the Lemma 1.

Proof of Theorem 4. The p.d.f of X is given as in (2.1), while we have,

$$P(X>t)=\int_t^\infty \phi(x)dx. \tag{2.12}$$

Using above lemma with $n=0$, it follows that

$$\int_t^\infty \phi(x)dx=\frac{\phi(t)}{t}-\int_t^\infty \frac{\phi(x)}{x^2}dx. \tag{2.13}$$

Using above lemma with $n=1$, it follows that

$$\int_t^\infty \frac{\phi(x)}{x^2}dx=\frac{\phi(t)}{t^3}-3\int_t^\infty \frac{\phi(x)}{x^4}dx. \tag{2.14}$$

From (2.13) and (2.14), we get

$$\int_t^\infty \phi(x)dx=\phi(t)\left(\frac{1}{t}-\frac{1}{t^3}\right)+3\int_t^\infty \frac{\phi(x)}{x^4}dx. \tag{2.15}$$

Again using above lemma with $n=2$, we obtain

$$\int_t^\infty \frac{\phi(x)}{x^4}dx=\frac{\phi(t)}{t^5}-5\int_t^\infty \frac{\phi(x)}{x^6}dx.$$

Hence from (2.15), we get

$$\int_t^\infty \phi(x)dx=\phi(t)\left(\frac{1}{t}-\frac{1}{t^3}+\frac{1\cdot 3}{t^5}\right)-(3\cdot 5)\int_t^\infty \frac{\phi(x)}{x^6}dx. \tag{2.16}$$

On proceeding in this way, a repeated application of the above lemma yields,

$$\begin{aligned}\int_t^\infty \phi(x)dx&=\phi(t)\left(\frac{1}{t}-\frac{1}{t^3}+\frac{1\cdot 3}{t^5}-\frac{1\cdot 3\cdot 5}{t^7}+\cdots\right),\\ &=\frac{1}{\sqrt{2\pi}}\,e^{-\frac{t^2}{2}}\left(\frac{1}{t}-\frac{1}{t^3}+\frac{1\cdot 3}{t^5}-\frac{1\cdot 3\cdot 5}{t^7}+\cdots\right),\end{aligned} \tag{2.17}$$

which in conjunction with (2.12) gives

$$P(X>t)=\frac{1}{\sqrt{2\pi}}\,e^{-\frac{t^2}{2}}\left(\frac{1}{t}-\frac{1}{t^3}+\frac{1\cdot 3}{t^5}-\frac{1\cdot 3\cdot 5}{t^7}+\cdots\right).$$

By symmetry

$$\begin{aligned}P(|X|>t)&=2P(X>t)\\ &=\sqrt{\frac{2}{\pi}}\,e^{-\frac{t^2}{2}}\left(\frac{1}{t}-\frac{1}{t^3}+\frac{1\cdot 3}{t^5}-\frac{1\cdot 3\cdot 5}{t^7}+\cdots\right).\end{aligned}$$

This completes the proof of Theorem 4.

Some deductions.

1. A repeated application of the lemma above shows that

$$\begin{aligned}0\le\int_t^\infty \frac{\phi(x)}{x^2}dx&=\frac{\phi(t)}{t^3}-3\int_t^\infty \frac{\phi(x)}{x^4}dx,\\ &=\phi(t)\left[\frac{1}{t^3}-\frac{3}{t^5}\right]+\int_t^\infty \frac{\phi(x)}{x^6}dx,\\ &\vdots\\ &=\phi(t)\left(\frac{1}{t^3}-\frac{1\cdot 3}{t^5}+\frac{1\cdot 3\cdot 5}{t^7}-\cdots\right),\end{aligned}$$

which implies,

$$\phi(t)\left(-\frac{1}{t^3}+\frac{1\cdot 3}{t^5}-\frac{1\cdot 3\cdot 5}{t^7}+\cdots\right)\le 0.$$

Using this in Theorem 4, we get

$$P(|X|>t)\le\sqrt{\frac{2}{\pi}}\,\frac{e^{-\frac{t^2}{2}}}{t},$$

which is Gaussian Tail inequality (GTI).

2. As before, a repeated application of the lemma above, shows that

$$0\le\int_t^\infty \frac{\phi(x)}{x^6}dx=\phi(t)\left(\frac{1\cdot 3\cdot 5}{t^7}-\frac{1\cdot 3\cdot 5\cdot 7}{t^9}+\cdots\right),$$

which gives,

$$\left(-\frac{1\cdot 3\cdot 5}{t^7}+\frac{1\cdot 3\cdot 5\cdot 7}{t^9}-\cdots\right)\le 0.$$

Hence from Theorem 4, we obtain,

$$\begin{aligned}P(|X|>t)&\le\sqrt{\frac{2}{\pi}}\,e^{-\frac{t^2}{2}}\left(\frac{1}{t}-\frac{1}{t^3}+\frac{1}{t^5}\right),\\ &=\sqrt{\frac{2}{\pi}}\,e^{-\frac{t^2}{2}}\left(\frac{t^4-t^2+3}{t^5}\right)\ \text{for } t>0,\end{aligned}$$

which is the conclusion of Theorem 2.

Many other interesting results can be deduced from Theorem 4 in a similar fashion.

3 CONCLUSION

These results of the authors can be used for finding better tail bounds for normal density. Certain results concerning the tail bound for normal distribution are obtained. These results refine known tail bound Gaussian Tail Inequality.

ACKNOWLEDGEMENTS

The authors are highly grateful to the referee whose commendable suggestions and comments added greatly to the quality of this paper.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

References

[1] Adler RJ, Muller P, Rozovski BL. Stochastic modelling in physical oceanography. Springer; 1996.

[2] Adler RJ, Taylor JE, Worsly KJ. Applications of random fields and geometry: Foundations and case studies. Springer; 2010.

[3] Borell C. The Brunn-Minkowski inequality in gauss space. Inventiones Mathematicae. 1975;30(2):207-216.

[4] Tsirelson B, Ibragimov I, Sudakov V. Norms of Gaussian sample functions. Proceedings of the third Japan-ussr symposium on probability theory (Tashkent, 1975). 1976;550:20-41.

[5] Piterbarg VI. Asymptotic methods in the theory of Gaussian processes and fields. Providence, R.I: American Mathematical Society; 1996.

[6] Sun JY. Tail probabilities of the maxima of Gaussian random fields. Annals of Probability. 1993;21(1):34-71.

[7] Azais JM, M Wschebor. A general expression for the distribution of the maximum of a Gaussian field and the approximation of the tail. Stochastic Processes and Their Applications .2008;118 (7):1190-1218.

[8] Adler R, Taylor J. Random Fields and Geometry. Springer; 2007.

[9] Liu J. Tail approximations of integrals of Gaussian random fields. Annals of Probability. 2012;40(3):1069-1104.

[10] Liu J, Xu G. Some asymptotic results of Gaussian random fields with varying mean functions and the associated. Annals of Statistics. 2012;40:262-293.

[11] Liu J, Xu G. On the conditional distributions and the efficient simulations of exponential integrals of Gaussian Random fields. Annals of Applied Probability to appear; 2013.
Available: http://arxiv.org/abs/1204.5546.

[12] Adler RJ, Blanchet, Liu JC. Efficient Monte Carlo for high excursions of Gaussian random Fields. The Annals of Applied Probability; 2012.

[13] Adler RJ, Blanchet, Liu JC. Fast simulation of Gaussian random fields. In Winter Simulation Conference; 2008.

[14] Azas JM, Wschebor M. Level sets and extrema of random processes and fields. John Wiley and Sons; 2009.

[15] Li X, Liu J. Rare event simulation and efficient discretization for the supremum of Gaussian random fields. Technical Report (arXiv: 1309.73652013).

[16] Liu J, Xu G. Rare event simulations for exponential integrals of smooth Gaussian processes. In Proceedings of the Winter Simulation Conference. Winter Simulation Conference; 2012.

[17] Liu J, Xu G. Efficient simulations for the exponential integrals of holder continuous Gaussian random fields. The ACM Transactions on Modeling and Computer Simulation. 2014;24(9):1-9.

[18] Gordon RD. Values of Milli's ratio of area to bounding ordinate and of the normal probability integral for large values of the argument. The Annals of Mathematical Statistics. 1941;12(3):364-366.

[19] Fan P. Stochastic process: Theory and applications. Press of I Singhua University; 2006.

Peer-review history:
The peer review history for this paper can be accessed here:
http://www.sciencedomain.org/review-history/26082