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Generalized Euclidean Least Square Approximation

A. S. Oke[1*], S. M. Akintewe[1] and A. G. Akinbande[1]

[1] *Department of Mathematical Sciences, Faculty of Science, Adekunle Ajasin University, Akungba Akoko, Ondo State, Nigeria.*

Authors' contributions

This work was carried out in collaboration between all authors. Author ASO derived the scheme and prepared the manuscript. Authors SMA and AGA jointly proved the existence and uniqueness theorem. All authors read and approved the final manuscript.

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ABSTRACT

A Generalised Euclidean Least Square Approximation (ELS) is derived in this paper. The Generalised Euclidean Least Square Approximation is derived by generalizing the interpolation of n arbitrary data set to approximate functions. Existence and uniqueness of the ELS schemes are shown by establishing the invertibility of the coefficient matrix using condensation method to reduce the matrix. The method is illustrated for exponential function and the results are compared to the classical Maclaurin's series.

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**Corresponding author: E-mail: abayomi.oke@aaua.edu.ng, okeabayomisamuel@gmail.com*

1 INTRODUCTION

In order to deal with some difficult functions, it is important to reduce such to a linear combination of some simpler functions. A way to reduce complicated functions is by approximation which involves finding a combination of some elementary functions to a desired degree of accuracy. Interpolation is another way to achieve this approximation but it involves fitting of functions to a given set of data and finding the best function in a certain class that can be used to represent the data. Interpolation and approximation become very significant because many problems in engineering and science require fitting a curve for a given set of experimental data or reduction of some nonlinear problems for easy manipulation. Some common methods include the Least-Square, Min-Max, Taylor series and Interpolatory Approximation (like the Lagrange and Newton) and many more. Many approximation methods have been employed in mathematics but Maclaurin series stand out in that it produces an elegant mathematical formula that requires less computation except for finding the derivatives of functions. In a study by [1], Euclidean space with their inner products was used to describe the methods of least squares adjustment as orthogonal projections on finite-dimensional subspaces and it was established that both the Euclidean and Hilbert space techniques in least squares adjustment are elegant and powerful geodetic methods. [2] made an analysis of euclidean distances and least squares problems for a given set of vectors, he proved that given an $m \times n$ real matrix $A = [a_1, ..., a_n]$, with $D(A) \equiv \text{diag}(\delta_i(A)) \quad (i = 1, \cdots, n)$, $\delta_i(A)$ is the Euclidean distance from a_i to the space spanned by all other columns of A i.e.

$$\delta_i(A) = \min_{x_i} \left\| a_i - [a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_n] \, x_i \right\|_2 = \left\| A\overline{x}_i \right\|_2.$$

Application of interpolation with finite element are considered by [3, 4]. [5] made an advancement of the momentum interpolation method on non-staggered grids. [6] developed a numerical method to quasi-interpolate the forcing term of differential equations by using radial basis functions. Least square approximations are of high significance in regression analysis for trimming outliers. [7] introduced a new methodology for identification of the parameters of the local linear Takagi-Sugeno fuzzy models using weighted recursive least squares (WRLS). [8] developed an algorithm for robust leverage to identify multiple outliers estimate based on Least Trimmed Euclidean Deviations (LTED). [9] developed the least square tracking algorithm which generalizes a suitable way for the standard LQR formulation on vector spaces to systems that evolve on the Euclidean group. The applications of least-squares include robotics, manipulating parts with complex shapes in unstructured dynamic environments, planning end-effect or motions around singularities, writing on a blackboard, and other motion planning problems involving kinematic constraints [9]. Other works involving the use of interpolation in Euclidean space can be found in [10, 11, 12, 13, 14, 15, 16]. [17] made an investigation of the effects of interpolation error, using two error methods. [18] derived the exact formulas for trigonometric sums at different nodes using Hermite interpolations whose nodes are basically zeros of Chebyshev polynomials of the first and second kinds.

In this paper, a generalized scheme for least square method called Euclidean Least Square Method (ELS) is derived by generalizing the interpolation of n arbitrary data set to approximate functions. Existence and uniqueness theorems are proved for the derived ELS schemes by establishing the invertibility of the coefficient matrix using condensation method to reduce the matrix. The method is illustrated for some transcendental functions and the results are compared to the classical Maclaurin's series. These schemes are useful for obtaining approximation to functions and also to fit a function for a set of data. The scheme avoids the difficulty that may arise in finding higher derivatives of functions.

2 DERIVATION OF THE ELS FORMULA

In order to derive the ELS schemes, the Euclidean norm for a given of $n+1$ data points $((x_i, y_i), \ i = 0\,(1)\,n)$ will be taken and the least square method will be used to minimize the residual. It is referred to as "generalized" because the scheme assumes an arbitrary order s.

Definition 2.1. Let $x \in \mathbb{R}^n$, then we define the Euclidean norm ([8]) as

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}.$$

Define the Euclidean Least Square (ELS) polynomial $P_s(x)$ of order s as

$$P_s(x) = \sum_{m=0}^{s} a_m x^m$$

for a given of $n+1$ data points $((x_i, y_i)\,,\ i = 0\,(1)\,n)$. Assume the exact function that produced the set is $y = f(x)$ and set $y_i = f\,(x_i)$ then define the deviation R_i of $P_s\,(x)$ from y at a point x_i and take the Euclidean norm

$$R_i = y_i - \sum_{m=0}^{s} a_m x_i^m \qquad \text{and so} \qquad \|R\|^2 = \sum_{i=0}^{n}\left(y_i - \sum_{m=0}^{s} a_m x_i^m\right)^2$$

Setting $\frac{\partial \left\|R\right\|^2}{\partial a_j} = 0$ to minimize $\left\|R\right\|^2$ and rearranging to get

$$\sum_{m=0}^{s}\left(a_m \sum_{i=0}^{n} x_k^{m+j}\right) = \sum_{i=0}^{n} x_i^j y_i, \quad \Rightarrow \quad \sum_{m=0}^{s} \alpha_m \alpha_{j+m} = \beta_j, \quad j = 0, 1, 2, \cdots, s. \tag{2.1}$$

where

$$\alpha_r = \sum_{i=0}^{n} x_i^r, \quad \beta_r = \sum_{i=0}^{n} x_i^r y_i. \tag{2.2}$$

Putting

$$\alpha = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_s \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{s+1} \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_s & \alpha_{s+1} & \alpha_{s+2} & \cdots & \alpha_{2s} \end{pmatrix},\ A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_s \end{pmatrix},\ \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_s \end{pmatrix}$$

We finally have

$$\alpha A = \beta \quad \Rightarrow \ A = \alpha^{-1}\beta. \tag{2.3}$$

Clearly, the coefficient matrix α is a symmetric $(s+1)\times(s+1)$ matrix and β is an $(s+1)$ column vector and a unique solution to equation 2.3 exists if $|\alpha| \neq 0$.

2.1 Existence and Uniqueness Theorem

Theorem 2.1. *The determinant of α is the the sum of exactly $(n+1)^{(s+1)}$ determinants of $(s+1) \times (s+1)$ matrices*

Proof. Define the determinant of α as

$$|\alpha| \quad = \quad \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_s \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{s+1} \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_s & \alpha_{s+1} & \alpha_{s+2} & \cdots & \alpha_{2s} \end{vmatrix}$$

and on substituting equation (2.2),

$$|\alpha| = \begin{vmatrix} \sum_{i=0}^{n} x_i^0 & \sum_{i=0}^{n} x_i^1 & \sum_{i=0}^{n} x_i^2 & \cdots & \sum_{i=0}^{n} x_i^s \\ \sum_{i=0}^{n} x_i^1 & \sum_{i=0}^{n} x_i^2 & \sum_{i=0}^{n} x_i^3 & \cdots & \sum_{i=0}^{n} x_i^{s+1} \\ \sum_{i=0}^{n} x_i^2 & \sum_{i=0}^{n} x_i^3 & \sum_{i=0}^{n} x_i^4 & \cdots & \sum_{i=0}^{n} x_i^{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^{n} x_i^s & \sum_{i=0}^{n} x_i^{s+1} & \sum_{i=0}^{n} x_i^{s+2} & \cdots & \sum_{i=0}^{n} x_i^{2s} \end{vmatrix}$$

and by condensation method gives

$$|\alpha| = \sum_{k_0=0}^{n} \begin{vmatrix} x_{k_0}^0 & \sum_{i=0}^{n} x_i^1 & \sum_{i=0}^{n} x_i^2 & \cdots & \sum_{i=0}^{n} x_i^s \\ x_{k_0}^1 & \sum_{i=0}^{n} x_i^2 & \sum_{i=0}^{n} x_i^3 & \cdots & \sum_{i=0}^{n} x_i^{s+1} \\ x_{k_0}^2 & \sum_{i=0}^{n} x_i^3 & \sum_{i=0}^{n} x_i^4 & \cdots & \sum_{i=0}^{n} x_i^{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k_0}^s & \sum_{i=0}^{n} x_i^{s+1} & \sum_{i=0}^{n} x_i^{s+2} & \cdots & \sum_{i=0}^{n} x_i^{2s} \end{vmatrix}.$$

and thus,

$$|\alpha| = \sum_{k_0=0}^{n} \chi_{k_0} \; k_0 = 0, 1, \cdots, n. \quad \Rightarrow \; \chi_{k_0} = \begin{vmatrix} x_{k_0}^0 & \sum_{i=0}^{n} x_i^1 & \sum_{i=0}^{n} x_i^2 & \cdots & \sum_{i=0}^{n} x_i^s \\ x_{k_0}^1 & \sum_{i=0}^{n} x_i^2 & \sum_{i=0}^{n} x_i^3 & \cdots & \sum_{i=0}^{n} x_i^{s+1} \\ x_{k_0}^2 & \sum_{i=0}^{n} x_i^3 & \sum_{i=0}^{n} x_i^4 & \cdots & \sum_{i=0}^{n} x_i^{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k_0}^s & \sum_{i=0}^{n} x_i^{s+1} & \sum_{i=0}^{n} x_i^{s+2} & \cdots & \sum_{i=0}^{n} x_i^{2s} \end{vmatrix}$$

Further condensation of χ_{k_0} gives

$$\chi_{k_0} = \sum_{k_1=0}^{n} \begin{vmatrix} x_{k_0}^0 & x_{k_1}^1 & \sum_{i=0}^{n} x_i^2 & \cdots & \sum_{i=0}^{n} x_i^s \\ x_{k_0}^1 & x_{k_1}^2 & \sum_{i=0}^{n} x_i^3 & \cdots & \sum_{i=0}^{n} x_i^{s+1} \\ x_{k_0}^2 & x_{k_1}^3 & \sum_{i=0}^{n} x_i^4 & \cdots & \sum_{i=0}^{n} x_i^{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k_0}^s & x_{k_1}^{s+1} & \sum_{i=0}^{n} x_i^{s+2} & \cdots & \sum_{i=0}^{n} x_i^{2s} \end{vmatrix} = \begin{vmatrix} x_{k_0}^0 & \sum_{i=0}^{n} x_i^1 & \sum_{i=0}^{n} x_i^2 & \cdots & \sum_{i=0}^{n} x_i^s \\ x_{k_0}^1 & \sum_{i=0}^{n} x_i^2 & \sum_{i=0}^{n} x_i^3 & \cdots & \sum_{i=0}^{n} x_i^{s+1} \\ x_{k_0}^2 & \sum_{i=0}^{n} x_i^3 & \sum_{i=0}^{n} x_i^4 & \cdots & \sum_{i=0}^{n} x_i^{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k_0}^s & \sum_{i=0}^{n} x_i^{s+1} & \sum_{i=0}^{n} x_i^{s+2} & \cdots & \sum_{i=0}^{n} x_i^{2s} \end{vmatrix} = \sum_{k_1=0}^{n} \chi_{k_1}\left[k_0\right]$$

where

$$\chi_{k_1}\left[k_0\right] = \begin{vmatrix} x_{k_0}^0 & x_{k_1}^1 & \sum_{i=0}^{n} x_i^2 & \cdots & \sum_{i=0}^{n} x_i^s \\ x_{k_0}^1 & x_{k_1}^2 & \sum_{i=0}^{n} x_i^3 & \cdots & \sum_{i=0}^{n} x_i^{s+1} \\ x_{k_0}^2 & x_{k_1}^3 & \sum_{i=0}^{n} x_i^4 & \cdots & \sum_{i=0}^{n} x_i^{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k_0}^s & x_{k_1}^{s+1} & \sum_{i=0}^{n} x_i^{s+2} & \cdots & \sum_{i=0}^{n} x_i^{2s} \end{vmatrix}$$

and therefore

$$|\alpha| = \sum_{k_0=0}^{n} \sum_{k_1=0}^{n} \chi_{k_1}\left[k_0\right].$$

Further condensation of χ_{k_1} gives

$$\chi_{k_1} = \sum_{k_2=0}^{n} \chi_{k_2}\left[k_0, k_1\right] \qquad \text{where} \quad \chi_{k_2}\left[k_0, k_1\right] = \begin{vmatrix} x_{k_0}^0 & x_{k_1}^1 & x_{k_2}^2 & \cdots & \sum_{i=0}^{n} x_i^s \\ x_{k_0}^1 & x_{k_1}^2 & x_{k_2}^2 & \cdots & \sum_{i=0}^{n} x_i^{s+1} \\ x_{k_0}^2 & x_{k_1}^3 & x_{k_2}^3 & \cdots & \sum_{i=0}^{n} x_i^{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k_0}^s & x_{k_1}^{s+1} & x_{k_2}^{s+1} & \cdots & \sum_{i=0}^{n} x_i^{2s} \end{vmatrix},$$

and

$$|\alpha| = \sum_{k_0=0}^{n}\sum_{k_1=0}^{n}\sum_{k_2=0}^{n}\chi_{k_2}\left[k_0,k_1\right].$$

Continuing in this manner and by setting

$$\chi_{k_s}\left[k_0,k_1,\cdots,k_{s-1}\right] = \begin{vmatrix} x_{k_0}^0 & x_{k_1}^1 & x_{k_2}^2 & \cdots & x_{k_s}^s \\ x_{k_0}^1 & x_{k_1}^2 & x_{k_2}^2 & \cdots & x_{k_s}^s \\ x_{k_0}^2 & x_{k_1}^3 & x_{k_2}^3 & \cdots & x_{k_s}^s \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k_0}^s & x_{k_1}^{s+1} & x_{k_2}^{s+1} & \cdots & x_{k_s}^s \end{vmatrix},$$

we have

$$|\alpha| = \sum_{k_0=0}^{n}\sum_{k_1=0}^{n}\sum_{k_2=0}^{n}\cdots\sum_{k_s=0}^{n}\chi_{k_s}\left[k_0,k_1,k_2,\cdots,k_{s-1}\right]$$

This a sum of an $(n+1)^{(s+1)}$ determinants and the proof is complete. □

Theorem 2.2. *At least*

$$(n+1)^{s+1} - \frac{(n+1)!}{(n-s)!}$$

determinants are zero.

Proof. From theorem 2.1, χ_{k_s} represents the sum of determinant of an $(n+1)^{(s+1)}$ matrices that will be equal to the determinant of α and

$$\chi_{k_s} = \begin{vmatrix} x_{k_0}^0 & x_{k_1}^1 & x_{k_2}^2 & \cdots & x_{k_s}^s \\ x_{k_0}^1 & x_{k_1}^2 & x_{k_2}^2 & \cdots & x_{k_s}^s \\ x_{k_0}^2 & x_{k_1}^3 & x_{k_2}^3 & \cdots & x_{k_s}^s \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{k_0}^s & x_{k_1}^{s+1} & x_{k_2}^{s+1} & \cdots & x_{k_s}^s \end{vmatrix} = 0.$$

The determinant of any $\chi_{k_i}(i=0,1,\cdots,s)$ will only be zero if any of the k_i's is repeated. Since there are $(s+1)$ k_i's $(i=0,1,\cdots,s)$ and each of the k_i's can take exactly one of $(n+1)$ values at a time (i.e. $k_i=0,1,\cdots,n$) and in order not to have a zero determinant, we avoid repeating the k_i's and the total number of ways to arrange $s+1$ k_i's from the $n+1$ choices without repeating any k_i is exactly ${}^{n+1}P_{s+1}$. Thus, there are exactly

$${}^{n+1}P_{s+1} = \frac{(n+1)!}{(n-s)!}$$

without repeated columns. Hence at least

$$(n+1)^{s+1} - \frac{(n+1)!}{(n-s)!} = (n+1)^{s+1} - {}^{n+1}p_{s+1} \text{ determinants are zero.}$$

□

Theorem 2.3 (Existence and Uniqueness Theorem)**.** *α is invertible.*

Proof. By hypothesis, the data (x_i, y_i) are not repeated and therefore there is no repeated column in the coefficient matrix α. Consider any two distinct columns p and q of A (say, $A_p$and A_q) such that

$$A_p = \begin{pmatrix} \alpha_{p-1} & \alpha_p & \cdots & \alpha_{p+s-1} \end{pmatrix}^T \quad \text{and } A_q = \begin{pmatrix} \alpha_{q-1} & \alpha_q & \cdots & a_{q+s-1} \end{pmatrix}^T$$

A linear combination of A_p and A_q gives

$$aA_p + bA_q = 0 \tag{2.4}$$

substituting valuesA_p and A_q into (2.4)we get

$$\begin{array}{rcl} a\begin{pmatrix} \alpha_{p-1} & \alpha_p & \cdots & \alpha_{p+s-1} \end{pmatrix}^T + b\begin{pmatrix} \alpha_{q-1} & \alpha_q & \cdots & \alpha_{q+s-1} \end{pmatrix}^T & = & \begin{pmatrix} 0 & 0 & \cdots & 0 \end{pmatrix}^T \\ \begin{pmatrix} a\alpha_{p-1} + b\alpha_{q-1} & a\alpha_p + b\alpha_q & \cdots & a\alpha_{p+s-1} + b\alpha_{q+s-1} \end{pmatrix}^T & = & \begin{pmatrix} 0 & 0 & \cdots & 0 \end{pmatrix}^T \\ \begin{pmatrix} \sum_{j=1}^n ax_j^{p-1} + bx_j^{q-1} & \sum_{j=1}^n ax_j^p + bx_j^q & \cdots & \sum_{j=1}^n ax_j^{p+s-1} + bx_j^{q+s-1} \end{pmatrix}^T & = & \begin{pmatrix} 0 & 0 & \cdots & 0 \end{pmatrix}^T \end{array}$$

and generally, we have

$$\sum_{j=1}^{n} ax_j^{p-1+k} + bx_j^{q-1+k} = 0 \Rightarrow ax_j^{p-1+k} + bx_j^{q-1+k} = 0. \quad \text{where } k = 0, 1, 2, \cdots, s$$

and hence $a = b = 0$ (since $p \neq q$). Thus, any two columns are linearly independent and the matrix α contain $(s+1)$ linearly independent columns (i.e. $rank\,(\alpha) = n$) therefore α is invertible. This invertibility of α verifies that there exists a unique solution of equation 2.3. ☐

3 NUMERICAL ILLUSTRATION OF ELS SCHEME

It is worth mentioning that the Maclaurin series requires the evaluation of higher derivatives up to the order of interest which ELS scheme does not require. In this section, Maclaurin series is compared with the ELS polynomials for exponential function. ELS scheme is further used to forecast power consumption based on the data obtained from Energy Information Administration of the United States of America.

Example 3.1. *Exponential function* $\exp(x)$.

The ELS scheme $P_4(x)$ of order 4 is given as

$$P_4(x) = \sum_{m=0}^{4} a_m x^m$$

and the coefficients a_m is obtained using the scheme 2.3

$$A = \alpha^{-1}\beta$$

where

$$\alpha = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\ \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \end{pmatrix}, \ A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \ \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

and

$$\alpha_r = \sum_{i=0}^{n} x_i^r,$$
$$\beta_r = \sum_{i=0}^{n} x_i^r y_i.$$

Choosing the nodes over the compact interval $[0,1]$ with the step size $h = \frac{1}{5} = 0.2$ as $x_i = \frac{i}{5}$, $i = 0(1)5$ gives $y_i = e^{x_i}$ and

$$\alpha_r = \sum_{i=0}^{5} \left(\frac{i}{5}\right)^r = 5^{-r} \sum_{i=0}^{5} i^r,$$
$$\beta_r = \sum_{i=0}^{5} \left(\frac{i}{5}\right)^r e^{x_i},$$

so that

$$\alpha_0 = 6.0000,\ \alpha_1 = 3.0000,\ \alpha_2 = 2.2000,\ \alpha_3 = 1.8000,\ \alpha_4 = 1.5664,$$
$$\alpha_5 = 1.4160,\ \alpha_6 = 1.3130,\ \alpha_7 = 1.2394,\ \alpha_8 = 1.1852,$$
$$\beta_0 = 10.4792,\ \beta_1 = 6.4330,\ \beta_2 = 5.0861, \beta_3 = 4.3566,\ \beta_4 = 3.9062.$$

The coefficient matrix α can therefore be written as

$$\alpha = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\ \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \end{pmatrix} = \begin{pmatrix} 6 & 3 & 2.2 & 1.8 & 1.5664 \\ 3 & 2.2 & 1.8 & 1.5664 & 1.416 \\ 2.2 & 1.8 & 1.5664 & 1.416 & 1.313 \\ 1.8 & 1.5664 & 1.416 & 1.313 & 1.2394 \\ 1.5664 & 1.416 & 1.313 & 1.2394 & 1.1852 \end{pmatrix}$$

and the inverse of the coefficient matrix as

$$\alpha^{-1} = \begin{pmatrix} 0.9960 & -9.0939 & 26.9097 & -31.8287 & 13.0208 \\ -9.0939 & 338.597 & -1548.03 & 2302.28 & -1085.07 \\ 26.9097 & -1548.03 & 7724.61 & -12026.2 & 5832.25 \\ -31.8287 & 2302.28 & -12026.2 & 19229.8 & -9494.36 \\ 13.0208 & -1085.07 & 5832.25 & -9494.36 & 4747.18 \end{pmatrix}.$$

The constant matrix β is

$$\beta = \begin{pmatrix} 10.4792 \\ 6.4330 \\ 5.0861 \\ 4.3566 \\ 3.9062 \end{pmatrix}$$

By substituting α^{-1} and β into

$$A = \alpha^{-1}\beta$$

the coefficient vector A is obtained as

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1.0000 \\ 0.9988 \\ 0.5097 \\ 0.1402 \\ 0.0695 \end{pmatrix}.$$

Hence, the ELS Scheme for exponential function of order 4 is given as

$$P_4 \quad = \quad 1 + 0.9988x + 0.5097x^2 + 0.1402x^3 + 0.0695x^4.$$

The first five terms of the Maclaurin series for an exponential function is

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}.$$

Taking the mean square error of the Maclaurin's series of order 4 over the interval $[0, 2]$ gives 0.00426933 whereas the mean square error of ELS4 over the interval $[0, 2]$ gives 0.00001368. Figure 1 below reveals that ELS4 agrees more with the exact graph than the Maclaurin series of order 4. It is worth mentioning that the choice of the nodes x_i is not unique and thus each time the nodes are modified, a new polynomial of the same order is obtained. Table 1 below shows some other ELS4 for the exponential function when x_i's are chosen over the interval $[0, 1]$ but with different step sizes.

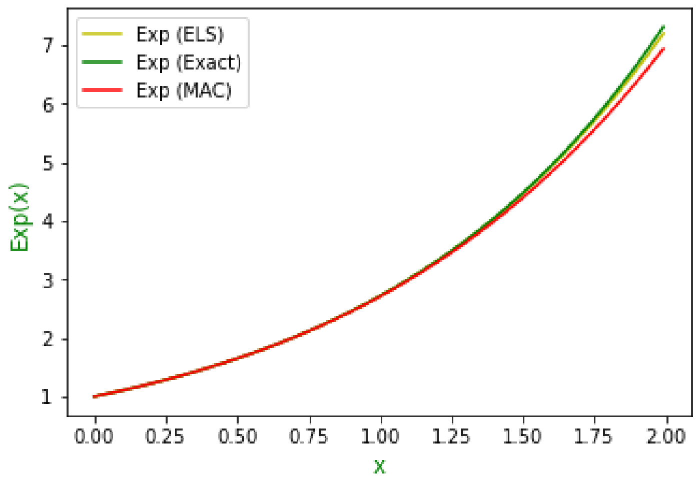

Fig. 1. Graph of an Exponential Function.

Table 1. Table showing different 4th order ELS approximation of the exponential function over$[0, 1]$

step size h	Mean Square Error ELS scheme	Mean Square Error Macluarin's series	$P_4(x)$
0.05	**0.00004262**	**0.00671770**	$1 + 0.9974x + 0.5166x^2 + 0.1290x^3 + 0.0752x^4$
0.1	**0.00003554**	**0.00585811**	$1 + 0.9981x + 0.5138x^2 + 0.1331x^3 + 0.0733x^4$
0.15	**0.00002554**	**0.00503930**	$1 + 0.9985x + 0.5115x^2 + 0.1368x^3 + 0.0714x^4$
0.2	**0.00001368**	**0.00426933**	$1 + 0.9988x + 0.5096x^2 + 0.1402x^3 + 0.0695x^4$
0.25	**0.000000000**	**0.00456362**	$1 + 0.9988x + 0.5097x^2 + 0.1402x^3 + 0.0694x^4$

Example 3.2. *Table 2 gives the recorded energy usage from the year 2006 to 2016 in the United States of America.*

Using the data from 2006 to 2015, ELS scheme of order 4 is obtained as

$$P_4(x) = 333.1343 + 24.3969x - 5.5276x^2 + 0.8039x^3 - 0.0391x^4.$$

This polynomial forecast for the year 2016 as 437.2333 which is pretty close to the exact value 440.1.

Table 2. Africa energy consumption between 2010 and 2016, data was extracted from Energy Information Administrative EIA https://www.bp.com/content/dam/bp/en/corporate/pdf/energy-economics/statistical-review-2017/bp-statistical-review-of-world-energy-2017-full-report.pdf

Africa Energy Consumption						
YEAR	2006	2007	2008	2009	2010	2011
Energy Consumption (thousand barrels) per day	334.8	347.9	369.5	373.4	388.9	388
YEAR	2012	2013	2014	2015	2016	
Energy Consumption (thousand barrels) per day	402.9	415.4	427.9	433.5	440.1	

4 CONCLUSIONS

A general formula for obtaining the Euclidean Least Square (ELS) scheme of arbitrary order is developed. The ELS polynomial $P_s(x)$ of order s is derived as

$$P_s(x) = \sum_{m=0}^{s} a_m x^m$$

where

$$A = \alpha^{-1}\beta,$$

$$\alpha = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_s \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{s+1} \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_s & \alpha_{s+1} & \alpha_{s+2} & \cdots & \alpha_{2s} \end{pmatrix}, A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_s \end{pmatrix}, \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_s \end{pmatrix}$$

and

$$\alpha_r = \sum_{i=0}^{n} x_i^r, \quad \beta_r = \sum_{i=0}^{n} x_i^r y_i.$$

The existence and uniqueness of the inverse of α suffices as a sufficient condition for the existence and uniqueness of ELS schemes. The schemes are used to approximate the exponential function. On comparing the resulting approximations with the classical Maclaurin's series, it is observed that the ELS schemes provide better approximations to this functions. It is worth noting that although the inversion of matrices may be deterrent to development of the ELS schemes, once the scheme is obtained, it is more accurate than the classical Maclaurin series. The scheme also has the advantage of flexible choice of the nodes in that the nodes can be adjusted until a desired accuracy is obtained.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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