An Efficient Third Derivative Hybrid Block Method of Order Eleven for the Direct Solution of Second-Order Ordinary Differential Equation

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Authors' contributions

This work was carried out in collaboration between all authors. Author DR designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors YS and TSK managed the analyses of the study. Author YS managed the literature searches. All authors read and approved the final manuscript.

Original Research Article

Abstract

This paper proposes an efficient third derivative hybrid block method of order eleven for solving second Order Ordinary Differential Equations directly. Method of interpolation and collocation of power series approximate solution to generate the continuous hybrid linear multistep method was used, which was evaluated at non-interpolating points to give a contnuous block method. The discrete block method was recovered when the continuous block method was evaluated at all step points. The basic properties of the method were investigated and were found to be zero-stable, consistent, and convergent. The efficiency of the method was tested on some stiff equations and was found to give better approximation than the existing method compared with our result.

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1 Introduction

This paper solves solution of stiff second order differential equation of the form

$$y'' = f(x, y(x), y'(x)), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0 \tag{1}$$

Where f is continuous within the interval of integration. Different method have been proposed for the solution of (1) ranging from predictor-corrector method to hybrid methods.despite the success recorded by the predictor-corrector methods, its major setback is that the predictor are in reducing order of accuracy especially when the value of the step-length is high and moreover the result are at overlapping interval. Direct methods of solving (1), which we shall employ has been discussed by many authors and they concluded that it is more convinient and accurate, among the authors that propose direct method are [1,2,3,4,5,6]. Block method have advantage of incorporating function evaluation at off-step points which afford the opportunity of circumventing the Dahlquist zero-stability barrier and it is actually possible to obtain convergent k-step methods of order 2k+1. Hence, hybrid block method is less expensive in terms of number of function evaluation compare to predictor-corrector method, it also possess the properties of Runge-kutta for being self-starting and does not require starting values. Other author who proposes block methods are [7,8,9,10,11,12].

In this paper, we developed a three-step third derivative hybrid block method for direct solution of second Order Ordinary differential equations, which is implemented in block method. The method developed evaluates less function per step and circumventing the Dahlquist barrier's by introducing a hybrid points.

The paper is organised as follows: In section 2, we discuss the methods and the materials for the development of the method. Section 3 considers analysis of the basis properties of the method, which include convergence and stability region, numerical experiments where the efficiency of the derived method is tested on some stiff numerical examples and discussion of results. Lastly, we concluded in section 4.

2 Derivation of the Method

We consider a power series approximate solution of the form

$$y(x) = \sum_{j=0}^{2s+r-1} a_i \left(\frac{x - x_n}{h} \right)^j \tag{2}$$

where $r = 2$ and $s = 5$ are the numbers of interpolation and collocation points respectively, is considered to be a solution to (1).

The second and third derivative of (2) gives

$$
\begin{aligned}
y''(x) &= \sum_{j=2}^{2s+r-1} \frac{a_j j!}{h^2(j-2)} \left(\frac{x - x_n}{h} \right)^{j-2} \\
&= f(x, y, y'), \\
y'''(x) &= \sum_{j=3}^{2s+r-1} \frac{a_j j!}{h^3(j-3)} \left(\frac{x - x_n}{h} \right)^{j-3} \\
&= g(x, y, y'),
\end{aligned}
\tag{3}
$$

Substituting (3) into (1) gives

$$f(x,y,y'') = \sum_{j=2}^{2s+r-1} \frac{a_j j!}{h^2(j-2)}\left(\frac{x-x_n}{h}\right)^{j-2} + \sum_{j=3}^{2s+r-1} \frac{a_j j!}{h^3(j-3)}\left(\frac{x-x_n}{h}\right)^{j-3} \tag{4}$$

Collocating (4) at all points $x_{n+s}, s = 0, \frac{1}{2}, 1, 2, 3$ and Interpolating Equation (2) at $x_{n+r}, r = 0, \frac{1}{2}$, gives a system of non linear equation of the form

$$AX = U \tag{5}$$

where

$$A = \begin{bmatrix} a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11} \end{bmatrix}^T$$

$$U = \begin{bmatrix} y_n, y_{n+\frac{1}{2}}, f_n, f_{n+\frac{1}{2}}, f_{n+1}, f_{n+2}, f_{n+3}, g_n, g_{n+\frac{1}{2}}, g_{n+1}, g_{n+2}, g_{n+3} \end{bmatrix}^T,$$

and

$$X = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \frac{1}{128} & \frac{1}{256} & \frac{1}{512} & \frac{1}{1024} & \frac{1}{2048} \\
0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2}{h^2} & \frac{3}{h^2} & \frac{3}{h^2} & \frac{5}{2h^2} & \frac{15}{8h^2} & \frac{21}{16h^2} & \frac{7}{8h^2} & \frac{9}{16h^2} & \frac{45}{128h^2} & \frac{55}{256h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} & \frac{30}{h^2} & \frac{42}{h^2} & \frac{56}{h^2} & \frac{72}{h^2} & \frac{90}{h^2} & \frac{110}{h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{12}{h^2} & \frac{24}{h^2} & \frac{40}{h^2} & \frac{60}{h^2} & \frac{84}{h^2} & \frac{112}{h^2} & \frac{144}{h^2} & \frac{180}{h^2} & \frac{220}{h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{18}{h^2} & \frac{36}{h^2} & \frac{60}{h^2} & \frac{90}{h^2} & \frac{126}{h^2} & \frac{168}{h^2} & \frac{216}{h^2} & \frac{270}{h^2} & \frac{330}{h^2} \\
0 & 0 & 0 & \frac{6}{h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{6}{h^3} & \frac{12}{h^3} & \frac{15}{h^3} & \frac{15}{h^3} & \frac{105}{8h^3} & \frac{21}{2h^3} & \frac{63}{8h^3} & \frac{45}{8h^3} & \frac{495}{128h^3} \\
0 & 0 & 0 & \frac{6}{h^3} & \frac{24}{h^3} & \frac{60}{h^3} & \frac{120}{h^3} & \frac{210}{h^3} & \frac{336}{h^3} & \frac{504}{h^3} & \frac{720}{h^3} & \frac{990}{h^3} \\
0 & 0 & 0 & \frac{6}{h^3} & \frac{48}{h^3} & \frac{120}{h^3} & \frac{240}{h^3} & \frac{420}{h^3} & \frac{672}{h^3} & \frac{1008}{h^3} & \frac{1440}{h^3} & \frac{1980}{h^3} \\
0 & 0 & 0 & \frac{6}{h^3} & \frac{72}{h^3} & \frac{180}{h^3} & \frac{360}{h^3} & \frac{630}{h^3} & \frac{1008}{h^3} & \frac{1512}{h^3} & \frac{2160}{h^3} & \frac{2970}{h^3}
\end{bmatrix}$$

Solving (5) for $a_i's$ using Gaussian elimination method, gives a continuous hybrid linear multistep method of the form

$$p(x) = \sum_{j=0,\frac{1}{2}} \alpha_j y_{n+j} + h^2 \left[\sum_{j=\frac{1}{2}} \beta_j f_{n+j} + \sum_{j=0}^{3} \beta_i f_{n+i} \right] + h^3 \left[\sum_{j=\frac{1}{2}} \gamma_j g_{n+j} + \sum_{i=0}^{3} \gamma_j g_{n+j} \right] \tag{6}$$

The coefficient of y_{n+j}, $j = 0, \frac{1}{2}$ and f_{n+j}, $j = 0, \frac{1}{2}, 1, 2, , 3$ gives

where

$$a_0 = \left(1 - \frac{2(x - x_n)}{h}\right), \quad a_{\frac{1}{2}} = \left(= \frac{2(x - x_n)}{h}\right)$$

$$\beta_0 = \frac{23}{2970}\frac{(x-x_n)^{11}}{h^9} - \frac{148}{1215}\frac{(x-x_n)^{10}}{h^8} + \frac{6307}{7776}\frac{(x-x_n)^9}{h^7} - \frac{18017}{6048}\frac{(x-x_n)^8}{h^6} + \frac{14939}{2268}\frac{(x-x_n)^7}{h^5} - \frac{14497}{1620}\frac{(x-x_n)^6}{h^4}$$
$$+ \frac{15563}{2160}\frac{(x-x_n)^5}{h^3} - \frac{139}{48}\frac{(x-x_n)^4}{h^2} + \frac{1}{2}(x-x_n)^2 - \frac{21654613}{153280512}(x-x_n)h$$

$$\beta_{\frac{1}{2}} = \frac{4096}{185625}\frac{(x-x_n)^{11}}{h^9} - \frac{9856}{30375}\frac{(x-x_n)^{10}}{h^8} + \frac{59776}{30375}\frac{(x-x_n)^9}{h^7} - \frac{149312}{23625}\frac{(x-x_n)^8}{h^6} + \frac{161792}{14175}\frac{(x-x_n)^7}{h^5} - \frac{565376}{50625}\frac{(x-x_n)^6}{h^4}$$
$$+ \frac{27392}{5625}\frac{(x-x_n)^5}{h^3} - \frac{256}{1125}\frac{(x-x_n)^4}{h^2} - \frac{2944883}{46777500}(x-x_n)h$$

$$\beta_1 = -\frac{3}{110}\frac{(x-x_n)^{11}}{h^9} + \frac{37}{90}\frac{(x-x_n)^{10}}{h^8} - \frac{743}{288}\frac{(x-x_n)^9}{h^7} + \frac{975}{112}\frac{(x-x_n)^8}{h^6} - \frac{2855}{168}\frac{(x-x_n)^7}{h^5} + \frac{287}{15}\frac{(x-x_n)^6}{h^4}$$
$$- \frac{231}{20}\frac{(x-x_n)^5}{h^3} + 3\frac{(x-x_n)^4}{h^2} - \frac{1255169}{28385280}(x-x_n)h$$

$$\beta_2 = -\frac{7}{2970}\frac{(x-x_n)^{11}}{h^9} + \frac{8}{243}\frac{(x-x_n)^{10}}{h^8} - \frac{1459}{7776}\frac{(x-x_n)^9}{h^7} + \frac{3401}{6048}\frac{(x-x_n)^8}{h^6} - \frac{2171}{2268}\frac{(x-x_n)^7}{h^5} + \frac{1519}{1620}\frac{(x-x_n)^6}{h^4}$$
$$- \frac{361}{720}\frac{(x-x_n)^5}{h^3} + \frac{17}{144}\frac{(x-x_n)^4}{h^2} - \frac{1139959}{766402560}(x-x_n)h$$

$$\beta_3 = -\frac{67}{371250}\frac{(x-x_n)^{11}}{h^9} + \frac{137}{60750}\frac{(x-x_n)^{10}}{h^8} - \frac{11207}{972000}\frac{(x-x_n)^9}{h^7} + \frac{11867}{378000}\frac{(x-x_n)^8}{h^6} - \frac{5611}{113400}\frac{(x-x_n)^7}{h^5}$$
$$+ \frac{4627}{101250}\frac{(x-x_n)^6}{h^4} - \frac{1579}{67500}\frac{(x-x_n)^5}{h^3} + \frac{2}{375}\frac{(x-x_n)^4}{h^2} - \frac{6136241}{95800320000}(x-x_n)h$$

$$\gamma_0 = \frac{1}{990}\frac{(x-x_n)^{11}}{h^8} - \frac{13}{810}\frac{(x-x_n)^{10}}{h^7} + \frac{281}{2592}\frac{(x-x_n)^9}{h^6} - \frac{205}{504}\frac{(x-x_n)^8}{h^5} + \frac{703}{756}\frac{(x-x_n)^7}{h^4} - \frac{361}{270}\frac{(x-x_n)^6}{h^3}$$
$$+ \frac{173}{144}\frac{(x-x_n)^5}{h^2} - \frac{23}{36}\frac{(x-x_n)^4}{h} + \frac{1}{6}(x-x_n)^3 - \frac{1751221}{255467520}(x-x_n)h^2$$

$$\gamma_{\frac{1}{2}} = \frac{128}{12375}\frac{(x-x_n)^{11}}{h^8} - \frac{64}{405}\frac{(x-x_n)^{10}}{h^7} + \frac{2048}{2025}\frac{(x-x_n)^9}{h^6} - \frac{5536}{1575}\frac{(x-x_n)^8}{h^5} + \frac{6784}{945}\frac{(x-x_n)^7}{h^4} - \frac{29248}{3375}\frac{(x-x_n)^6}{h^3}$$
$$+ \frac{2176}{375}\frac{(x-x_n)^5}{h^2} - \frac{128}{75}\frac{(x-x_n)^4}{h} + \frac{51643}{1559250}(x-x_n)h^2$$

$$\gamma_1 = \frac{1}{110}\frac{(x-x_n)^{11}}{h^8} - \frac{2}{15}\frac{(x-x_n)^{10}}{h^7} + \frac{233}{288}\frac{(x-x_n)^9}{h^6} - \frac{587}{224}\frac{(x-x_n)^8}{h^5} + \frac{39}{8}\frac{(x-x_n)^7}{h^4} - \frac{125}{24}\frac{(x-x_n)^6}{h^3} + 3\frac{(x-x_n)^5}{h^2}$$
$$- \frac{3}{4}\frac{(x-x_n)^4}{h} + \frac{59057}{5677056}(x-x_n)h^2$$

$$\gamma_2 = \frac{1}{990}\frac{(x-x_n)^{11}}{h^8} - \frac{11}{810}\frac{(x-x_n)^{10}}{h^7} + \frac{193}{2592}\frac{(x-x_n)^9}{h^6} - \frac{31}{144}\frac{(x-x_n)^8}{h^5} + \frac{269}{756}\frac{(x-x_n)^7}{h^4} - \frac{46}{135}\frac{(x-x_n)^6}{h^3}$$

$$+ \frac{43}{240}\frac{(x-x_n)^5}{h^2} - \frac{1}{24}\frac{(x-x_n)^4}{h} + \frac{131671}{255467520}(x-x_n)h^2$$

$$\gamma_3 = \frac{1}{24750}\frac{(x-x_n)^{11}}{h^8} - \frac{1}{2025}\frac{(x-x_n)^{10}}{h^7} + \frac{161}{64800}\frac{(x-x_n)^9}{h^6} - \frac{337}{50400}\frac{(x-x_n)^8}{h^5} + \frac{79}{7560}\frac{(x-x_n)^7}{h^4} - \frac{259}{27000}\frac{(x-x_n)^6}{h^3}$$

$$+ \frac{11}{2250}\frac{(x-x_n)^5}{h^2} - \frac{1}{900}\frac{(x-x_n)^4}{h} + \frac{12119}{912384000}(x-x_n)h^2$$

Where $t = \dfrac{x-x_n}{h}$, $y_{n+j} = y(x_n + jh)$ and $f_{n+j} = f\big((x_n + jh), y(x_n + jh), y'(x_n + jh)\big)$

Differentiating (6) once yields

$$p'(x) = \frac{1}{h}\sum_{j=0,\frac{1}{2}}\alpha_j y_{n+j} + h\left[\sum_{j=\frac{1}{2}}\beta_j f_{n+j} + \sum_{j=0}^{3}\beta_j f_{n+j}\right] + h^2\left[\sum_{j=\frac{1}{2}}\gamma_j g_{n+j} + \sum_{j=0}^{3}\gamma_j g_{n+j}\right] \qquad (7)$$

Equation (6) is evaluated at non-interpolating points, and evaluating (7) at all points gives a discrete block formula of the form

$$A^{(0)}Y_m^{(i)} = \sum_{i=0}^{1} h^i e_i y_n^{(i)} + h^2 b_i f(y_n) + h^2 d_i f(Y_m) + h^3 c_i g(y_n) + h^3 r_i g(Y_m) \qquad (8)$$

Where

$$Y_m = \left[y_{n+\frac{1}{2}}, y_{n+1}, y_{n+2}, y_{n+3}\right]^T \quad, \quad f(y_m) = \left[f_{n+\frac{1}{2}}, f_{n+1}, f_{n+2}, f_{n+3}\right]^T \quad, \quad g(y_m) = \left[g_{n+\frac{1}{2}}, g_{n+1}, g_{n+2}, g_{n+3}\right]^T$$

$$y_n^{(i)} = \left[y_{n-1}^{(i)}, y_{n-2}^{(i)}, y_{n-3}^{(i)}, y_n^{(i)}\right]^T \quad, \quad f(y_n) = [f_{n-1}, f_{n-2}, f_{n-3}, f_n]^T \quad, \quad g(y_n) = [g_{n-1}, g_{n-2}, g_{n-3}, g_n]^T$$

And $A^{(0)} = 8 \times 8$ identity matrix.

when i = 0

$$e_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad, \quad e_i = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad, \quad b_0 = \begin{bmatrix} 0 & 0 & 0 & \frac{21654613}{306561024} \\ 0 & 0 & 0 & \frac{41257}{249480} \\ 0 & 0 & 0 & \frac{5399}{13365} \\ 0 & 0 & 0 & \frac{1233}{1232} \end{bmatrix} \quad, \quad d_0 = \begin{bmatrix} \frac{2944883}{93555000} & \frac{1255169}{56770560} & \frac{1139959}{1532805120} & \frac{6136241}{1916006400\,00} \\ \frac{36416}{155925} & \frac{243}{2464} & \frac{589}{249480} & \frac{493}{4989600} \\ \frac{1650688}{1670625} & \frac{248}{495} & \frac{1411}{13365} & \frac{2312}{1670625} \\ \frac{159552}{48125} & -\frac{7047}{12320} & \frac{4167}{6160} & \frac{122211}{1540000} \end{bmatrix}$$

$$c_0 = \begin{bmatrix} 0 & 0 & 0 & \frac{1751221}{510935040} \\ 0 & 0 & 0 & \frac{137}{15840} \\ 0 & 0 & 0 & \frac{778}{31185} \\ 0 & 0 & 0 & \frac{1053}{12320} \end{bmatrix} \quad, \quad r_0 = \begin{bmatrix} -\frac{51643}{3118500} & -\frac{59057}{11354112} & -\frac{131671}{510935040} & -\frac{12119}{1824768000} \\ -\frac{763}{17325} & -\frac{529}{27720} & \frac{269}{332640} & \frac{17}{831600} \\ -\frac{2048}{779625} & -\frac{404}{3465} & -\frac{652}{31185} & -\frac{212}{779625} \\ \frac{864}{1375} & \frac{1215}{1232} & \frac{1647}{12320} & \frac{801}{77000} \end{bmatrix}$$

when i = 1

$$
e_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, b_0 = \begin{bmatrix} 0 & 0 & 0 & \dfrac{12976613}{69672960} \\ 0 & 0 & 0 & \dfrac{52363}{272160} \\ 0 & 0 & 0 & \dfrac{2497}{8505} \\ 0 & 0 & 0 & \dfrac{1161}{1120} \end{bmatrix}, \quad d_0 = \begin{bmatrix} \dfrac{245039}{1063125} & \dfrac{207791}{2580480} & \dfrac{182827}{69672960} & \dfrac{975887}{8709120000} \\ \dfrac{506752}{1063125} & \dfrac{659}{659} & \dfrac{1061}{272160} & \dfrac{5311}{34020000} \\ \dfrac{1063125}{45056} & \dfrac{2016}{76} & \dfrac{272160}{685} & \dfrac{124}{124} \\ \dfrac{42525}{18048} & \dfrac{315}{3321} & \dfrac{1701}{519} & \dfrac{42525}{47589} \\ \dfrac{4375}{} & -\dfrac{}{1120} & \dfrac{}{1120} & \dfrac{}{140000} \end{bmatrix}
$$

$$
c_0 = \begin{bmatrix} 0 & 0 & 0 & \dfrac{234319}{23224320} \\ 0 & 0 & 0 & \dfrac{977}{90720} \\ 0 & 0 & 0 & \dfrac{13}{567} \\ 0 & 0 & 0 & \dfrac{129}{1120} \end{bmatrix}, \quad r_0 = \begin{bmatrix} -\dfrac{37879}{567000} & -\dfrac{139}{3360} & -\dfrac{3007}{3317760} & -\dfrac{13481}{580608000} \\ \dfrac{2944}{70875} & -\dfrac{139}{3360} & -\dfrac{17}{12960} & \dfrac{73}{2268000} \\ \dfrac{2048}{14175} & \dfrac{3360}{105} & -\dfrac{23}{405} & -\dfrac{8}{14175} \\ \dfrac{1152}{875} & \dfrac{1863}{1120} & \dfrac{63}{160} & \dfrac{921}{28000} \end{bmatrix}
$$

3 Analysis of Basic Properties of the Method

3.1 Order of the block

Let the linear operator $L\{y(x) : h\}$ associated with the discrete block method (8) be defined

$$
L\{y(x) : h\} = A^{(0)}Y_m^i - \sum_{j=0}^{3} h^i e_i y_n^i - h^2(b_0 f(y_n) + d_0 F(Y_m)) - h^3(c_0 g(y_n) + G(Y_m)) \tag{9}
$$

Expanding (9) in Taylor series and comparing the coefficient of h gives

$$
L\{y(x) : h\} = C_0 y(x) + C_1 y'(x) + \ldots + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) + C_{p+2} h^{p+2} y^{p+2}(x) + \ldots
$$

Definition: Linear operator L and associated block formula are said to be of order

$p, if C_0 = C_1 = \ldots = C_p = C_{p+1} = 0, \ and \ C_{p+2} \neq 0.$ C_{p+2} called the error constant and implies that the truncation error is given by $t_{n+k} = C_{p+2} h^{p+2} y^{p+2}(x) + O(h^{p+3})$

For our method, expanding (8) in Taylor series

and comparing the coefficient of h gives $C_0 = C_1 = C_2 = C_3 = \ldots = C_{11} = 0$ and

$$
C_{12} = \left[\frac{238807}{206009008128000}, \frac{233}{67060224000}, \frac{13}{449064000}, \frac{87}{275968000}, \frac{51643}{12875563008000}, \frac{529}{100590336000}, \frac{163}{3143448000}, \frac{89}{137984000} \right]^T
$$

3.2 Zero stability of our method

Definition: A block method is said to be zero-stable if as $h \to 0$, the root $z_i, i = 1(1)k$ of the first characteristic polynomial $\rho(z) = 0$ that is $\rho(z) = \det\left[\sum_{j=0}^{k} A^{(i)} z^{k-i}\right] = 0$ Satisfies $|z_i| \leq 1$ and for those roots with $|z_i| = 1$, multiplicity must not exceed two. The block method for k=3, with one off grid collocation point expressed in the form

$$
\rho(z) = z \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{h}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2h \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = z^6 (z-1)^2
$$

$$\rho(z) = z^6 (z-1)^2 = 0,$$

Hence, our method is zero-stable.

3.3 Region of absolute stability of the two step one offstep point

We shall adopt the boundary locus method to determine the region of absolute stability of the implicit three-step third derivative hybrid block method. This gives stability polynomial below

$$
\begin{aligned}
\overline{h}(w) = {} & h^{12}\left(\left(\frac{8581}{299376000000}\right)w^4 - \left(\frac{23329}{34927200000}\right)w^3\right) + h^{11}\left(\left(\frac{969161}{4191264000000}\right)w^4 + \left(\frac{25354921}{1047816000000}\right)w^3\right) \\
& - h^{10}\left(\left(\frac{13481}{93139200000}\right)w^4 + \left(\frac{349245913}{838252800000}\right)w^3\right) + h^9\left(\left(\frac{20731903}{1077753600000}\right)w^4 + \left(\frac{172960967}{1077753600000}\right)w^3\right) \\
& - h^8\left(\left(\frac{177004147}{1077753600000}\right)w^4 + \left(\frac{22083914003}{1077753600000}\right)w^3\right) + h^7\left(\left(\frac{464273}{907200000}\right)w^4 - \left(\frac{653723467}{39916800000}\right)w^3\right) \\
& + h^6\left(\left(\frac{2144917}{1995840000}\right)w^4 + \left(\frac{306556939}{1995840000}\right)w^3\right) - h^5\left(\left(\frac{19800841}{1496880000}\right)w^4 + \left(\frac{420829301}{598752000}\right)w^3\right) \\
& + h^4\left(\left(\frac{4569569}{166320000}\right)w^4 - \left(\frac{74560469}{166320000}\right)w^3\right) + h^3\left(\left(\frac{7457}{111375}\right)w^4 - \left(\frac{673693}{356400}\right)w^3\right) \\
& - h^2\left(\left(\frac{561383}{1782000}\right)w^4 + \left(\frac{7457617}{1782000}\right)w^3\right) + w^4 - w^3
\end{aligned}
$$

The absolute stability region of the new method is plotted and shown below
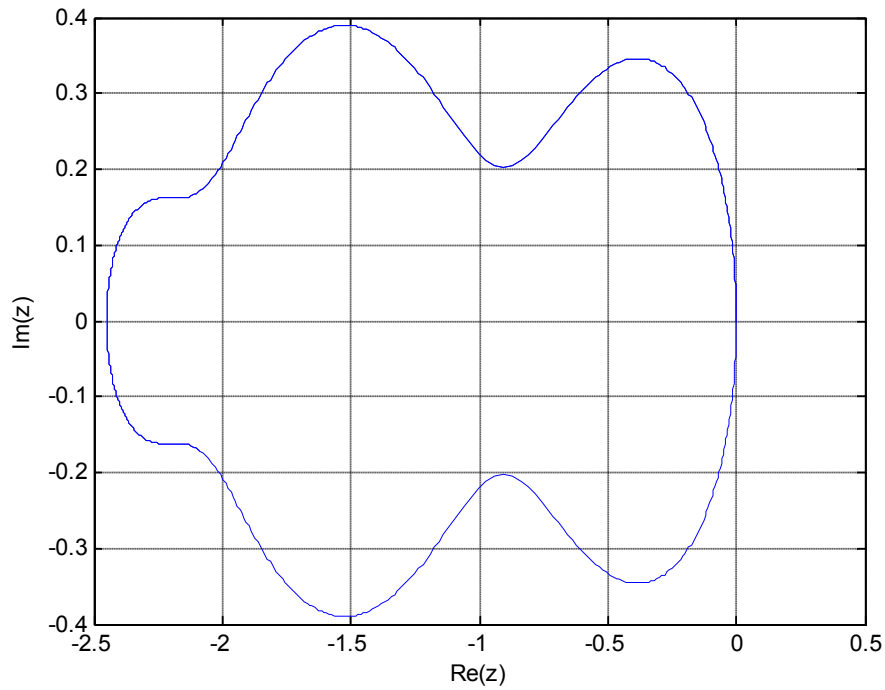
3.4 Numerical example

Problem I. We consider a highly stiff problem

$y''+1001y'+1000y$, $\quad y(0)=1, y'(0)=-1$

Exact Solution: $y(x)=\exp(-x), h=\dfrac{1}{10}$

x-values	Exact solution	Computed solution	Error in our method	Error in [8]
0.100	0.90483741803595957316	0.90483741803595957382	6.600000E(-19)	1.054712E(-14)
0.200	0.81873075307798185867	0.81873075307798185402	4.650000E(-18)	1.776357E(-14)
0.300	0.74081822068171786607	0.74081822068171786407	2.000000E(-18)	2.342571E(-14)
0.400	0.67032004603563930074	0.67032004603563929888	1.860000E(-18)	2.797762E(-14)
0.500	0.60653065971263342360	0.60653065971263342181	1.790000E(-18)	3.130829E(-14)
0.600	0.54881163609402643263	0.54881163609402643074	1.890000E(-18)	3.397282E(-14)
0.700	0.49658530379140951470	0.49658530379140951247	2.230000E(-18)	3.563816E(-14)
0.800	0.44932896411722159143	0.44932896411722158881	2.620000E(-18)	3.674838E(-14)
0.900	0.40656965974059911188	0.40656965974059910872	3.160000E(-18)	3.730349E(-14)
1.00	0.36787944117144232160	0.36787944117144231770	3.900000E(-18)	3.741452E(-14)

Problem II. $f(x, y, y')= y'$, $\quad y(0)= 0, y'(0)=-1$.

Exact Solution: $y(x)=1- e^x$ $\;with, h=\dfrac{1}{100}$

x-values	Exact solution	Computed solution	Error in our method	Error in [10]
0.01	-0.1051709180756476248	-0.10517091807564762481	1.00000E(-20)	8.32667E(-17)
0.02	-0.2214027581601698339	-0.22140275816016983388	2.00000E(-20)	2.775558(-16)
0.03	-0.3498588075760031040	-0.34985880757600310361	3.90000E(-19)	5.551116(-16)
0.04	-0.4918246976412703178	-0.49182469764127031664	1.16000E(-18)	9.436896(-16)
0.05	-0.6487212707001281468	-0.64872127070012814473	2.07000E(-18)	2.109425(-15)
0.06	-0.8221188003905089749	-0.82211880039050897131	3.59000E(-18)	3.219647(-15)
0.07	-1.0137527074704765216	-1.01375270747047651580	5.80000E(-18)	4.440892(-15)
0.08	-1.2255409284924676046	-1.22554092849246759630	8.30000E(-18)	5.995204(-15)
0.09	-1.4596031111569496638	-1.45960311115694965230	1.15000E(-17)	7.771561(-15)
0.10	-1.7182818284590452354	-1.71828182845904521940	1.60000E(-17)	1.065814(-14)

4 Conclusions

It is evident from the above tables that our proposed methods is indeed accurate, and can handle stiff equations. Also in terms of stability analysis, the method is $A-stable$.

Comparing the new method with the existing method [8,10],the result presented in the tables 1 and 2 shows that the new method performs better than the existing method [8,10]. In this article, a three-step block method with one off-step point is derived via the interpolation and collocation approach. The developed method is consistent, convergent and zero-stable with an A-Stable region and order eleven.

Competing Interests

Authors have declared that no competing interests exist.

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