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αIg-Separation Axioms in Ideal Topological Spaces

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Authors' contributions

This work was carried out in collaboration between both authors. Author DV designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author SM managed the analyses of the study and the literature searches. Both authors read and approved the final manuscript.

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Abstract

In this research paper, αIg-open sets are used to define and study some separation axioms in ideal topological spaces. The implications of these axioms among themselves and with the known axioms are investigated.

Keywords: Ideal topological spaces; αIg-closed sets; αIg-T_0 space; αIg-T_1 space; αIg-T_3 space.

1 Introduction

The subject of ideals in topological spaces has been introduced by Kuratowski [1] and Vaidyanathasamy [2]. An Ideal I on a topological space (X, τ) is defined as a non-empty collection I of subsets of X satisfying the following two conditions (i) if $A \in I$ and $B \subset A$, then $B \in I$ (ii) If $A \in I$ and $B \in I$,then $A \cup B \in I$. Given a topological space (X, τ) with an ideal I on X and if P(X) is the set of all subsets of X, a set operator $(.)^*$: P(X)→P(X), called the local function [2] of A with respect to τ and I, is defined as follows : For $A \subset X$, $A^*(\tau,I) = \{x \in X / U \cap A \in I$ for every open neighbourhood U of x$\}$. A Kuratowski closure operator $cl^*(.)$ for a

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topology $\tau^*(\tau,I)$ called the *-topology, finer than τ is defined by $cl^*(A) = A \cup A^*(\tau,I)$ where there is no chance of confusion, $A^*(I)$ is denoted by A^*. If I is an ideal on X, then (X,τ,I) is called an ideal topological space. In this paper, αIg-closed sets are used to define some weak separation axioms and to study some of their basic properties. The implications of these axioms among themselves and with the known axioms are investigated.

Separation axioms on topological spaces are those to classify the classes of topological spaces. T_2 axiom is an important axiom as it has many applications. Several topologists [3,4,5] concentrate on separation axioms between T_0, T_1, and T_2. In this paper, the concept of αIg-T_0 space, αIg-T_1 space, and αIg-T_3 space are introduced , characterized and studied their relationships with some of known axioms.

2 Preliminaries

Definition 2.1[6]: Let (X, τ) be a topological space and I be an ideal on X. A subset A of X is said to be α-Ideal generalized closed set (αIg-closed set) if $A^* \subseteq U$ whenever $A \subseteq U$ and U is α-open.

Definition 2.2[7]: A subset A of a topological space (X,τ) is said to be clopen, if it is both open and closed in (X,τ).

Definition 2.3[8]: A topological space (X, τ) is said to be T_0 space if for each pair of distinct points x, y of X, there exists an open set containing one point but not the other.

Definition 2.4[7]: A topological space (X, τ) is said to be T_1 space if for each pair of distinct points x, y of X, there exists a pair of open sets, one containing x but not y and the other containing y but not x.

Definition 2.5: A topological space (X, τ) is said to be T_2 Space if for each pair of distinct points x, y of X, there exists open sets U and V such that $x \in U$ and $y \in V$ and $U \cap V = \phi$.

Definition 2.6[6]: A topological space (X, τ) is said to be Ultra Hausdroff space if for pair of distinct points x and y in X there exit two clopen sets U and V containing x and y such that $U \cap V = \phi$.

3 αIg-T_0 Spaces

In this section, an αIg-closed sets are used to define the topological space αIg-T_0 space and some of their properties are discussed.

Definition 3.1: An ideal topological space (X,τ,I) is said to be αIg-T_0 space if for each pair of distinct points x,y of X, there exists an αIg-open set containing one point but not the other.

Theorem 3.2: An ideal topological space (X,τ,I) is an αIg-T_0 space if and only if αIg-closures of distinct points are distinct.

Proof: Let x and y be two distinct points in X and X be an αIg-T_0 space. Then, there exists an αIg-open set G such that $x \in G$ but $y \notin G$. Also $x \notin G^c$ and $y \in G^c$ where G^c is an αIg-closed set in X. Since $\alpha Igcl(\{y\})$ is the intersection of all αIg-closed sets which contains y, $y \in \alpha Igcl(\{y\})$ but $x \notin \alpha Igcl(\{y\})$ as $x \notin G^c$. Thus,$\alpha Igcl(\{x\}) \neq \alpha Igcl(\{y\})$.

Conversely, suppose that for any pair of distinct points x and y in X, $\alpha Igcl(\{x\}) \neq \alpha Igcl(\{y\})$. Then, there exists atleast one point $z \in X$ such that $z \in \alpha Igcl(\{x\})$ but $z \notin \alpha Igcl(\{y\})$. If $x \in \alpha Igcl(\{y\})$, $\alpha Igcl(\{x\}) \subset \alpha Igcl(\{y\})$, then $z \in \alpha Igcl(\{y\})$, which is a contradiction. Hence $x \notin \alpha Igcl(\{y\})$. Now,$x \notin \alpha Igcl(\{y\})$ implies $x \in (\alpha Igcl(\{y\}))^c$, which is an αIg-open set in X containing x but not y. Hence X is an αIg-T_0 space.

Theorem 3.3: Every subspace of an αIg-T_0 space is an αIg-T_0 space.

Proof: Let X be anαIg-T_0 space and Y be a subspace of X. Let x,y be two distinct points of Y. Since $Y \subseteq X$ and X is an αIg-T_0 space, there exists an αIg-open set G such that $x \in G$ but $y \notin G$. Then, there exists an αIg-open set $G \cap Y$ in Y which contains x but does not contain y. Hence Y is an αIg-T_0 space.

Theorem 3.4: Every T_0 space is an αIg-T_0 space.

Proof: Let x and y be two distinct points in (X,τ,I) and X be an T_0 space. Then, there exists an open set G such that $x \in G$ and $y \notin G$. Since every open set is an αIg-open set, G is an αIg-open set where $x \in G$ and $y \notin G$. This implies, (X,τ,I) is an αIg-T_0 space.

Remark 3.5: The converse of the above theorem need not be true as seen from the following example.

Example 3.6: Consider the ideal topological space (X,τ,I), where $X=\{a,b,c\}$ with $\tau= \{\phi,\{a, b\},X\}$ and $I= \{\phi,\{a\}\}$.Then, X is an αIg-T_0 space but not T_0 space, since a and b are contained in all the open sets of X.

Definition 3.7: A function f: $(X,\tau,I) \rightarrow (Y,\sigma,J)$ is said to beαIg-totally continuous, if the inverse image of every αIg-open subset of Y is clopen in X.

Theorem 3.8: Let f : $(X,\tau,I) \rightarrow (Y,\sigma,J)$ be an injective map and Y is an αIg-T_0 space. If f is an αIg-totally continuous then, X is Ultra-Hausdroff.

Proof: Let x and y be two distinct points in X. Since f is injective, f(x) and f(y) $\in$ Y such that $f(x) \neq f(y)$. Since Y is an αIg-T_0 space, there exists an αIg-open set U containing f(a) but not f(b). Then, we have a $\in f^{-1}$(U) and b $\notin f^{-1}$ (U). Thus,a $\in f^{-1}$(U),b $\in$ $(f^{-1}(U))^c$ and f^{-1} (U) is clopen in X because f is αIg-totally continuous. This implies, every pair of distinct points of X can be separated by disjoint clopen sets in X. Therefore, X is Ultra-Hausdroff.

Theorem 3.9: Let : $(X,\tau,I) \rightarrow (Y,\sigma,J)$ be an αIg-irresolute bijective map. If Y is an αIg-T_0 space, then X is αIg-T_0 space.

Proof: Assume that Y is an αIg-T_0 space. Let u, v be two distinct points of Y. Since f is a bijection, for every x,y$\in$ X such that f^{-1} (u) = x and f^{-1} (v)= y. Since Y is anαIg-T_0 space, there exists an αIg-open set H in Y such that u$\in$ H but v$\notin$ H. Since f is αIg-irresolute, f^{-1}(H) is αIg-open in X containing f(x) = u but not containing f(y) = v. Thus, there exists an αIg-open setf^{-1}(H) in X such that x $\in f^{-1}$(H) but y $\notin f^{-1}$(H) and hence X is an αIg-T_0 space.

4 αIg-T_1 Space

In this section, an αIg-closed sets are used to define the new topological space αIg-T_1 space and some of their properties are discussed.

Definition 4.1: An ideal topological space (X,τ,I) is said to be αIg-T_1 space, if for each pair of distinct points x, y of X, there exists a pair of αIg-open sets, one containing x but not y and the other containing y but not x.

Theorem 4.2: Every subspace of an αIg-T_1 space is also an αIg-T_1 space.

Proof: Let X be an αIg-T_1 space and let Y be a subspace of X.Let x, y $\in Y \subseteq X$ such that $x \neq y$. By hypothesis X is anαIg-T_1 space, hence there exists an αIg-open sets U, V in X such that x $\in$ U and y $\in$ V,

$x \notin V$ and $y \notin U$. By definition of subspace, $U \cap Y$ and $V \cap Y$ are αIg-open sets in Y. Further, $x \in U$, $x \in Y$ implies $x \in U \cap Y$ also $y \in V$, $y \in Y$ implies $y \in V \cap Y$. Thus, there exists αIg-open sets $U \cap Y$ and $V \cap Y$ in Y such that $x \in U \cap Y$, $y \in V \cap Y$ and $x \notin V \cap Y$, $y \notin U \cap Y$. Therefore, Y is an αIg-T_1 space.

Theorem 4.3: Every T_1 space is an αIg – T_1 space.

Proof: Let x and y be two distinct points in (X,τ,I) and X be an T_1 space. Then, there exists a pair of open sets U and V in X such that $x \in U$ and $y \notin U$, $y \in V$ and $x \notin V$. Since every open set is an αIg-open set, therefore U and V are αIg-open sets where $x \in U$ and $y \notin U$, $y \in V$ and $x \notin V$. This implies that, (X,τ,I) is an αIg-T_1 space.

Remark 4.4: The converse of the above theorem need not be true as seen from the following example

Example 4.5: Consider the ideal topological space (X,τ,I), where X={a,b,c} with τ= {ϕ,{a,b}, X} and I= {ϕ,{a}}.Then, X is anαIg-T_1 space but not T_1 space, since there is no open set containing a but not containing b.

Theorem 4.6: Every αIg-T_1 space is an αIg-T_0 space.

Proof: Suppose X is an αIg-T_1space, then for distinct points x and y in X, there exists an αIg-open sets G and H such that $x \in G$, $y \notin G$ and $y \in H$, $x \notin H$. Since $G \cap H = \phi$, $x \in G$ and $y \in H$. Then, either $x \in G$, $y \notin G$ or $y \in H$, $x \notin H$. Thus, X is an αIg-T_0 space.

Remark 4.7: The converse of the above theorem need not be true as seen from the following example.

Example 4.8: Consider the ideal topological space (X,τ,I), where X = {a,b,c} with τ = {ϕ,{a,b}, X} and I = {ϕ,{a}}.Then, X is an αIg-T_0 space but not an αIg-T_1 space since for the distinct points b and c, there exists a pair of αIg-open sets {a,b} and {b,c} one containing b but not c and the other containing both b and c.

Theorem 4.9: Let f: (X,τ,I) → (Y,σ,J) be an injective and Y be an αIg-T_1 space. If f is αIg-irresolute, then X is an αIg-T_1 space.

Proof: Assume that Y is an αIg-T_1 space. Let $x,y \in Y$ such that $x \neq y$. Then, there exists a pair of αIg-open sets U,V in Y such that $f(x) \in U$ and $f(y) \in V$, $f(x) \notin V$ and $f(y) \notin U$ which implies $x \in f^{-1}(U)$, $y \in f^{-1}(V)$ and $x \notin f^{-1}(V)$, $y \notin f^{-1}(U)$. Since f is αIg-irresolute, X is αIg-T_1 space.

Theorem 4.10: If f: (X,τ,I) → (Y,σ,J) is αIg-totally continuous and Y is αIg-T_1 space, then X is clopen.

Proof: Let x and y be any two distinct points in X. Since f is injective, $f(x)$ and $f(y) \in Y$ such that $f(x) \neq f(y)$. Since Y is anαIg-T_1 space, there exists αIg-open sets U and V in Y such that $f(x) \in U$ and $f(y) \notin U$, $f(y) \in V$ and $f(x) \notin V$. Therefore, we have $x \in f^{-1}(U)$ and $y \notin f^{-1}(U)$, $y \in f^{-1}(V)$ and $x \notin f^{-1}(V)$, where $f^{-1}(U)$ and $f^{-1}(V)$ are clopen subsets of X since$f$ is αIg-totally continuous function. This shows that, X is clopen.

Theorem 4.11: If {x} is αIg-closed in X, for every $x \in X$,then X is αIg-T_1 space.

Proof: Let x,y be two distinct points of X such that {x} and {y} are αIg-closed. Then, $\{x\}^c$ and $\{y\}^c$ areαIg-open in X such that $y \in \{x\}^c$ but $x \notin \{x\}^c$ and $x \in \{y\}^c$ but $y \notin \{y\}^c$. Hence X is an αIg-T_1 space.

5 αIg – T_2 Space

In this section, an αIg-closed sets are used to define the new topological space αIg-T_2 space and some of their properties are discussed.

Definition 5.1: An ideal topological space (X,τ,I) is said to be αIg-T_2space, if for each pair of distinct points x, y of X, there exists αIg-open sets U and V such that $x \in U$ and $y \in V$ and $U \cap V = \phi$.

Theorem 5.2: Every T_2 Space is an αIg-T_2 Space.

Proof: Let x and y be two distinct points in (X,τ,I) and X be an T_2 space. Then, there exists a pair of open set U, V in X such that $x \in U$ and $y \in V$ and $U \cap V = \phi$.Since every open set is an αIg-open set, therefore U and V are αIg-open sets where $x \in U$ and $y \in V$ and $U \cap V = \phi$. This implies (X,τ,I) is an αIg-T_2 space.

Remark 5.3: The converse of the above theorem need not be true as seen from the following example.

Example 5.4: Consider the ideal topological space (X,τ,I), where X= {a,b,c,d} with τ={ϕ,{a},{b,d},{a,b,d}, X} and I = {ϕ, {a}}. Then, X is an αIg-T_2 Space but not T_2 Space because the intersection of open sets { } and {a,b,d} is not empty.

Theorem 5.5: Every αIg-T_2space is an αIg-T_1space.

Proof: Suppose X is anαIg-T_2 Space, then for distinct points x and y in X there exists αIg-open sets G and H such that $G \cap H = \phi$. Therefore, $x \in G$, $y \notin G$ and $y \in H$, $x \notin H$. Thus, X is an αIg – T_1 space.

Remark 5.6: The converse of the above theorem need not be true as seen from the following example.

Example 5.7: Consider the ideal topological space (X,τ,I), where X= {a,b,c} with τ={ϕ,{a},{a,c}, X} and I = {ϕ,{a}}. Then, X is an αIg-T_1space but not an αIg-T_2space because the intersection of αIg-open sets {a,b} and {a,c} is not empty.

Theorem 5.8: Every subspace of an αIg-T_2 Space is also anαIg-T_2 Space.

Proof: Let X be anαIg-T_2 space and let Y be a subspace of X. Let $a, b \in Y \subseteq X$ with $a \neq b$. By hypothesis, there exists αIg-open sets G,H in X such that $a \in G$ and $b \in H$, $G \cap H = \phi$. By definition of subspace, $G \cap Y$ and $H \cap Y$ are αIg-open sets in Y. Further $a \in G$, $a \in Y$ implies $a \in G \cap Y$ and $b \in H$, $b \in Y$ implies $b \in H \cap Y$. Since $G \cap H = \phi$, $(Y \cap G) \cap (Y \cap H) = Y \cap (G \cap H) = Y \cap \phi = \phi$. Therefore, $G \cap Y$ and $H \cap Y$ are disjoint αIg-open sets in Y such that $a \in G \cap Y$ and $b \in H \cap Y$. Thus, Y is αIg-T_2space.

Theorem 5.9: Iff: (X,τ,I) → (Y,σ,J) is αIg-totally continuous injection and Y is αIg-T_2 Space, then X is ultra-Hausdorff.

Proof: Let x and y be any two distinct points in X. Since f is injective, f(x) and $f(y) \in Y$ such that $f(x) \neq f(y)$. Since Y is an αIg-T_2space, there exists αIg-open sets U and V such that $f(x) \in U$ and $f(y) \in V$ and $U \cap V = \phi$. This implies, $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. Since f is αIg-totally continuous, f^{-1}(U) and f^{-1}V) are clopen sets in X. Also,$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \phi$. Thus, every two distinct points of X can be separated by disjoint clopen sets. Hence X is ultra-Hausdorff.

Theorem 5.10: If {x} is αI- closed in X, for every $x \in X$, then X is αIg-T_2 space.

Proof: Let x, y be two distinct points of X such that {x} and {y} are αIg-closed. Then, $\{x\}^c$ and $\{y\}^c$ areαIg-open in X such that y ∈ $\{x\}^c$ but x ∉ $\{x\}^c$ and x ∈ $\{y\}^c$ but y ∉ $\{y\}^c$.This implies, $\{x\}^c \cap \{y\}^c = \phi$. Hence X is αIg-$T_2$ space.

Theorem 5.11: If X is αIg-T_2 space, then for y≠x ∈X, there exists an αIg-open set G such that x ∈ G and y ∉αIg –cl(G).

Proof: Let x , y ∈ X such that y ≠ x. Since X is an αIg-T_2 space, there exists disjoint αIg- open sets G and H in X such that x ∈ G and y ∈ H. Therefore, H^c is αIg-closed set such that αIgcl(G) ⊆H^c. Since y ∈ H, we have y ∉$H^{c+.}$ Hence y ∉αIgcl(G).

Definition 5.12: Afunction f:(X, τ, I) → (Y, σ, J) is called totally αIg-continuous, if f^{-1}(V) is αIg-clopen in (X, τ, I) for each open set V in (Y, σ).

Theorem 5.13: If f: (X,τ,I) → (Y,σ,J) is totally αIg-continuous, injection and Y is T_0, then X is an αIg-T_2 space.

Proof: Let x and y be any two distinct points in X. Since f is injective, we have f(x) and f(y) ∈ Y such that f(x) ≠f (y). Since Y is T_0, there exists open set U containing f(x) but not f (y). Then, x ∈f^{-1}(U) and y ∉f^{-1}(U). Since f is totally αIg-continuous, f^{-1}(U) is an αIg-clopen subset of X. Also, x ∈f^{-1}(U) and y ∈ $(f^{-1}(U))^c$. Therefore, X is an αIg-T_2 space.

Theorem 5.14: A function f: X → Y is αIg-totally continuous, if its graph function is αIg-totally continuous.

Proof: Let g: X →X ×Y be the graph function of$f$: X → Y. Suppose g is αIg-totally continuous and F be anαIg-open set in Y. Then, X × F is an αIg-open set in X × Y. Since g is αIg-totally continuous, g^{-1} (X × F)= f^{-1}(F) is clopen in X. Thus, the inverse image of every αIg-open set in Y is clopen in X. Hence,f is αIg-totally continuous.

Theorem 5.15: Product of two αIg-T_0 space is also an αIg-T_0 space.

Proof: Let X and Y be two ideal topological spaces and let X ×Y be their product space. If x and y are distinct points of X, there exists an αIg-open set U in X such that it contains only one of these two and not the other, since X is an αIg-T_0 space. Let (x_1, y_1) and (x_2, y_2) be any two distinct points of X ×Y then either $x_1 \neq x_2$ or $y_1 \neq y_2$. If $x_1 \neq x_2$, there exists an αIg-open set U such that $x_1 \in$ U and $x_2 \notin$ U, since X is αIg-T_0 space. Therefore, U ×,Y is anαIg-open set containing (x_1, y_1) but not containing (x_2, y_2). Similarly, if$y_1 \neq y_2$, there exists an αIg-open set V such that $y_1 \in$ V and $y_2 \notin$ V, since Y is an αIg-T_0 space. Therefore, X ×V is anαIg-open set containing (x_1, y_1) but not containing (x_2, y_2).Hence corresponding to distinct points of X × Y, there exists anαIg-open set containing one but not the other so that X × Y is anαIg-T_0 space.

Theorem 5.16: Product of two αIg-T_1 space is also an αIg-T_1 space.

Proof: Let X and Y be two ideal topological spaces and let X× Y be their product space. Let (x,y) be an arbitrary point of X × Y so that x ∈X and y ∈Y. Since X and Y are αIg-T_1 space,{x} and {y} are αIg-closed in X and Y respectively and hence x∈ X^c and y∈ Y^c are αIg-open. Then, (x,y)∈ $(X \times Y)^c$ is anαIg-open set. Thus, {(x,y)} is αIg-closed.

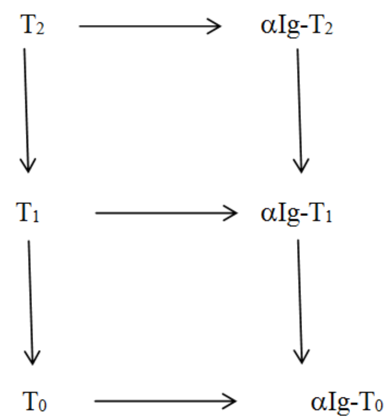
Theorem 5.17: Product of two αIg-T_2 space is an αIg-T_2 space.

Proof: Let X and Y be two ideal topological spaces and let X × Y be their product space.Let x and y be distinct points of X. Let (x_1, y_1) and (x_2, y_2) be any two distinct points of X×Y then, either $x_1 \neq x_2$ or $y_1 \neq y_2$.

If $x_1 \neq x_2$ and since X is αIg-T_2 space, there exists two αIg-open sets U and V in X such that $x_1 \in U, x_2 \in V$ and $U \cap V = \phi$. Hence $U \times Y$ and $V \times Y$ are αIg-open sets containing (x_1, y_1) and (x_2, y_2) respectively such that $(U \times Y) \cap (V \times Y) = (U \cap V) \times Y = \phi$. Hence $(X \times Y)$ is anαIg-T_2 space.

6 Diagram

As a consequence of the theorems [3.4,4.3,4.6,5.2,5.5] and remarks [3.5,4.4,4.7,5.3,5.6] the following implication diagram holds.



In this diagram, A → B means A implies B but does not imply A.

7 Conclusion

The concept of αIg-T_0 space, αIg-T_1 space, and αIg-T_3 space were introduced, characterized and studied their relationships with some of known axioms.

Competing Interests

Authors have declared that no competing interests exist.

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