Journal of Advances in Mathematics and Computer Science

29(4): 1-15, 2018; Article no.JAMCS.16241

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Approximate Best Proximity Pairs in Fuzzy Normed Spaces for Contraction Maps

S. A. M. Mohsenailhosseini[1,2*]

[1]*Faculty of Mathematics, Vali-e-Asr University, Rafsanjan, Iran.*
[2]*Faculty of Mathematics, Yazd University, Yazd, Iran.*

Author's contribution

The sole author analysed, designed, interpreted and prepared the whole manuscript.

Article Information

DOI: 10.9734/JAMCS/2018/16241
Editor(s):
(1) Dr. Heng-you Lan, Professor, Department of Mathematics, Sichuan University of Science & Engineering, China.
Reviewers:
(1) Choonkil Park, Hanyang University, South Korea.
(2) Elida Hoxha, University of Tirana, Albania.
(3) Chengjun Guo, School of Applied Mathematics, Guangdong University of Technology, China.
(4) Asha Rani, SRM University, India.
Complete Peer review History: http://www.sciencedomain.org/review-history/27082

Received: 02 January 2015
Accepted: 09 March 2015
Published: 06 November 2018

Original Research Article

Abstract

We define approximate best proximity pair and diameter approximate best proximity pair in fuzzy normed spaces. Two general lemmas are given regarding approximate best proximity pair of operators on fuzzy normed spaces. Using these results we prove theorems for various types of well known generalized contractionson on fuzzy normed spaces.

Keywords: Fuzzy normed spasc; fuzzy approximate best proximity pair; diameter fuzzy approximate best proximity pair.

2010 Mathematics Subject Classification: 46A32, 46M05, 41A17

**Corresponding author: E-mail: amah@vru.ac.ir; mohsenhosseini@yazd.ac.ir; mohsenialhosseini@gmail.com*

1 Introduction

Fuzzy set was defined by Zadeh [1] and Katsaras [2], who while studying fuzzy topological vector spaces, was the first to introduce in 1984 the idea of fuzzy norm on a linear space. In 1992, Felbin [3] defined a fuzzy norm on a linear space with an associated metric of the Kaleva and Seikkala type [4]. A further development along this line of inquiry took place when in 1994, Cheng and Mordeson [5] evolved the definition of a further type of fuzzy norm having a corresponding metric of the Kramosil and Michalek type [6].

Chitra and Mordeson [7] introduced the definition of fuzzy norm and thereafter the concept of fuzzy normed space has been introduced and generalized in different ways by Bag and Samanta in [8], [9], [10]. In 2011, Mohsenalhosseini et al. [11], introduced the approximate fixed point in fuzzy normed spaces.

In this paper starting from the article of Mohsenalhosseini and Mazaheri [12], we study some well known types of operators on fuzzy normed spaces, and we give some qualitative and quantitative results regarding approximate best proximity pairs of such operators.

Throughout this article, the symbols $\wedge$ and $\vee$ mean the min and the max, respectively.

2 Some Preliminary Results

We begin by recalling some needed definitions and results.

Definition 2.1. *Let U be a linear space on $\boldsymbol{R}$. A function $N : U \times \boldsymbol{R} \to [0,1]$ is called fuzzy norm if and only if for every $x, u \in U$ and for every $c \in \boldsymbol{R}$ the following properties are satisfied :*

(F_{N1}) : $N(x,t) = 0$ *for every* $t \in \boldsymbol{R}^- \cup \{0\}$,

(F_{N2}) : $N(x,t) = 1$ *if and only if* $x = 0$ *for every* $t \in \boldsymbol{R}^+$,

(F_{N3}) : $N(cx,t) = N(x, \frac{t}{|c|})$ *for every* $c \neq 0$ *and* $t \in \boldsymbol{R}^+$,

(F_{N4}) : $N(x+u, s+t) \geq min\{N(x,s), N(u,t)\}$ *for every* $s, t \in \boldsymbol{R}^+$,

(F_{N5}) : *the function* $N(x,.)$ *is nondecreasing on* $\boldsymbol{R}$, *and* $\lim_{t\to\infty} N(x,t) = 1$.

The pair (U, N) is called a fuzzy normed space. Sometimes, We need two additional conditions as follows:

(F_{N6}) : $\forall t \in \mathbf{R}^+ \; N(x,t) > 0 \; \Rightarrow x = 0$.

(F_{N7}) : function $N(x,.)$ is continuous for every $x \neq 0$, and on subset

$$\{t : \; 0 < N(x,t) < 1\}$$

is strictly increasing.

Let (U, N) be a fuzzy normed space. For all $\alpha \in (0,1)$, we need define α norm on U as follows :

$$\|x\|_\alpha = \wedge\{t > 0 : N(x,t) \geq \alpha\} \; for \; every \; x \in U.$$

Then $\{\|x\|_\alpha : \; \alpha \in (0,1]\}$ is an ascending family of normed on U and they are called of $\alpha-$norm on U. We give some notation, lemmas and example which will be used in this paper.

Lemma 2.1. *[8] Let (U, N) be a fuzzy normed space such that satisfy conditions F_{N6} and F_{N7}. Define the function $N' : U \times \boldsymbol{R} \to [0,1]$ as follows:*

$$N'(x,t) = \begin{cases} \vee\{\alpha \in (0,1) : \|x\|_\alpha \leq t\} & (x,t) \neq (0,0) \\ 0 & (x,t) = (0,0) \end{cases}$$

Then
a)N' *is a fuzzy norm on* U*.*
b) $N = N'$*.*

Lemma 2.2. *[8] Let* (U, N) *be a fuzzy normed space such that satisfy conditions* F_{N6} *and* F_{N7}*. and* $\{x_n\} \subseteq U$*, Then* $\lim_{n\to\infty} N(x_n - x, t) = 1$ *if and only if*

$$\lim_{n\to\infty} \|x_n - x\|_\alpha = 0$$

for every $\alpha \in (0, 1)$*.*

Note that the sequence $\{x_n\} \subseteq U$ *converges if there exists* $x \in U$ *such that*

$$\lim_{n\to\infty} N(x_n - x, t) = 1 \ for \ every \ t \in \boldsymbol{R}^+.$$

In this case x *is called the limit of* $\{x_n\}$*.*

Definition 2.2. [13] Let (U, N) be a fuzzy normed space, $T : U \to U$, $\epsilon > 0$ and $u_0 \in U$. Then u_0 is a F^z_ϵ-approximate fixed point (fuzzy approximate fixed point) of T if for some $\alpha \in (0, 1)$

$$\wedge\{t > 0 : N(u_0 - Tu_0, t) \geq \alpha\} \leq \epsilon.$$

Remark 2.1. [13] In this paper we will denote the set of all F^z_ϵ-approximate fixed points (fuzzy approximate fixed points) of T, for a given $\epsilon > 0$, by

$$F^z_\epsilon(T) = \{u \in U : \wedge\{t > 0 : N(u - Tu, t) \geq \alpha\} \leq \epsilon \ for \ some \ \alpha \in (0, 1)\}.$$

Definition 2.3. [13] Let $T : U \to U$. Then T has the fuzzy approximate fixed point property (f.a.f.p.p.) if

$$\forall \ \epsilon > 0 \ F^z_\epsilon(T) \neq \emptyset.$$

3 Fuzzy Approximate Best Proximity Pair for Various Types of Operators

In the section, we begin with two lemmas which will be used in order to prove all the results given in the same section.

Definition 3.1. Let (U, N) be a fuzzy normed space which satisfies conditions F_{N6} and F_{N7} and $\{\|.\|_\alpha : \alpha \in (0, 1)\}$ the set of $\alpha-$norms defined on U. Suppose A and B are nonempty subsets of U, $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$ and $\epsilon > 0$. Then $u_0 \in A \cup B$ is a F^z-approximate best proximity points of the pair (A, B), if for some $\alpha \in (0, 1)$

$$\wedge\{t > 0 : N(u_0 - Tu_0, t) \geq \alpha\} \leq \wedge\{t > 0 : N(A - B, t) \geq \alpha\} + \epsilon.$$

Remark 3.1. *In the rest of the paper we will denote the set of all* F^z*-approximate best proximity point of the pair* (A, B)*, for a given* $\epsilon > 0$*, by*

$$F^z_\epsilon(A, B) = \{u_0 \in A \cup B : \wedge\{t > 0 : N(u_0 - Tu_0, t) \geq \alpha\} \leq \wedge\{t > 0 : N(A - B, t) \geq \alpha\} + \epsilon\}$$

for some $\alpha \in (0, 1)$.

Lemma 3.1. *Let (U, N) be a fuzzy normed space which satisfies conditions F_{N6} and F_{N7}, and A and B be nonempty subsets of U. Suppose that the mapping $T : A\cup B \to A\cup B$ is satisfies $T(A) \subseteq B$, $T(B) \subseteq A$ and for some $\alpha \in (0, 1)$,*

$$Lim_{n\to\infty} \wedge \{t > 0 : N(T^n u - T^{n+1}u, t) \geq \alpha\} = \wedge\{t > 0 : N(A - B, t) \geq \alpha\}, \text{ for some } u \in A \cup B.$$

Then the pair (A, B) is a fuzzy approximate best proximity pair.

Proof: *Let $\epsilon > 0$ be given and $u \in A \cup B$ such that*

$$Lim_{n\to\infty} \wedge \{t > 0 : N(T^n u - T^{n+1}u, t) \geq \alpha\} = \wedge\{t > 0 : N(A - B, t) \geq \alpha\}.$$

Then there exists $N_0 > 0$ such that

$$\forall n \geq N_0 : \quad \wedge \{t > 0 : N(T^n u - T^{n+1}u, t) \geq \alpha\} < \wedge\{t > 0 : N(A - B, t) \geq \alpha\} + \epsilon$$

for some $\alpha \in (0, 1)$. If $n = N_0$, then for some $\alpha \in (0, 1)$

$$\wedge\{t > 0 : N(T^{N_0} u - T(T^{N_0})u, t) \geq \alpha\} < \wedge\{t > 0 : N(A - B, t) \geq \alpha\} + \epsilon.$$

Therefore $T^{N_0}u \in F^z_\epsilon(A, B)$ and $F^z_\epsilon(A, B) \neq \emptyset$. Hence there exists a fuzzy approximate best proximity pair in $A \cup B$. ■

Definition 3.2. Let (U, N) be a fuzzy normed space such that satisfy conditions F_{N6} and $F_{N7,}$ and A and B be nonempty subsets of U. We define fuzzy diameter of $F^z_\epsilon(A, B) \neq \varnothing$ for some $\alpha \in (0, 1)$ as

$$\delta^\alpha(F^z_\epsilon(A, B)) = \vee[\wedge\{t > 0 : N(u - v, t) \geq \alpha \ \forall \ u, v \in U\}].$$

Lemma 3.2. *Let (U, N) be a fuzzy normed space such that satisfy conditions F_{N6} and F_{N7}, and A and B be nonempty subsets of U. Suppose that the mapping $T : A \cup B \to A \cup B$ is satisfies $T(A) \subseteq B$, $T(B) \subseteq A$ and $\epsilon > 0$. We assume that for some $\alpha \in (0, 1)$:*

1) $F^z_\epsilon(A, B) \neq \varnothing$;

2) for all $\mu > 0$ there exists $\varphi(\mu) > 0$ such that

$$\wedge\{t > 0 : N(u - v, t) \geq \alpha\} - \wedge\{t > 0 : N(Tu - Tv, t) \geq \alpha\} \leq \mu$$
$$\Rightarrow \wedge\{t > 0 : N(u - v, t) \geq \alpha\} \leq \varphi(\mu) \ \forall u, v \in F^z_\epsilon(A, B).$$

Then:

$$\delta^\alpha(F^z_\epsilon(A, B)) \leq \varphi(2(\wedge\{t > 0 : N(A - B, t) \geq \alpha\} + \epsilon)).$$

Proof: *Let $\epsilon_1, \epsilon_2 > 0$ and $u, v \in F^z_\epsilon(A, B)$. Then:*

$$\wedge\{t > 0 : N(u - Tu, t) \geq \alpha\} \leq \wedge\{t > 0 : N(A - B, t) \geq \alpha\} + \epsilon_1$$

and

$$\wedge\{t > 0 : N(v - Tv, t) \geq \alpha\} \leq \wedge\{t > 0 : N(A - B, t) \geq \alpha\} + \epsilon_2.$$

We can write:

$$\wedge\{t > 0 : N(u - v, t) \geq \alpha\} \leq \wedge\{t > 0 : N(u - Tu, t) \geq \alpha\} + \wedge\{t > 0 : N(Tu - Tv, t) \geq \alpha\}$$

$$+\wedge\{t > 0 : N(v-Tv, t) \geq \alpha\} \leq \wedge\{t > 0 : N(Tu-Tv, t) \geq \alpha\}+2(\wedge\{t > 0 : N(A-B, t) \geq \alpha\})+\epsilon_1+\epsilon_2$$

Put $\epsilon = Max\{\epsilon_1, \epsilon_2\}$, therfore,

$$\wedge\{t > 0 : N(u - v, t) \geq \alpha\} - \wedge\{t > 0 : N(Tu - Tv, t) \geq \alpha\} \leq 2(\wedge\{t > 0 : N(A - B, t) \geq \alpha\} + \epsilon).$$

Now by (2) it follow that

$$\wedge\{t > 0 : N(u - v, t) \geq \alpha\} \leq \varphi(2(\wedge\{t > 0 : N(A - B, t) \geq \alpha\} + \epsilon),$$

So

$$\delta^\alpha(F^z_\epsilon(A, B)) \leq \varphi(2(\wedge\{t > 0 : N(A - B, t) \geq \alpha\} + \epsilon)$$

for some $\alpha \in (0, 1)$. ■

Lemma 3.3. *Let (U, N) be a fuzzy normed space such that satisfy conditions F_{N6} and F_{N7}, and A and B be nonempty subsets of U. Suppose that the mapping $T : A \cup B \to A \cup B$ is satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and $\epsilon > 0$. We assume that for some $\alpha \in (0, 1)$,*

1) $Lim_{n\to\infty} \wedge \{t > 0 : N(T^n u - T^{n+1}u, t) \geq \alpha\} = \wedge\{t > 0 : N(A - B, t) \geq \alpha\}$, for some $u \in A \cup B$.

2) for all $\mu > 0$ there exists $\varphi(\mu) > 0$ such that

$$\wedge\{t > 0 : N(u - v, t) \geq \alpha\} - \wedge\{t > 0 : N(Tu - Tv, t) \geq \alpha\} \leq \mu$$
$$\Rightarrow \wedge\{t > 0 : N(u - v, t) \geq \alpha\} \leq \varphi(\mu) \ \forall u, v \in F_\epsilon^z(A, B).$$

Then:

$$\delta^\alpha(F_\epsilon^z(A, B)) \leq \varphi(2(\wedge\{t > 0 : N(A - B, t) \geq \alpha\}) + \epsilon).$$

Definition 3.3. Let (U, N) be a fuzzy normed space and A and B be nonempty subsets of U. A mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a F-P-a-contraction if there exists $a \in (0, 1)$ such that

$$\wedge\{t > 0 : N(Tu - Tv, t) \geq \alpha\} \leq a \wedge \{t > 0 : N(u - v, t) \geq \alpha\}, \ \forall u, v \in A \cup B.$$

Theorem 3.4. *Let (U, N) be a fuzzy norm spaceed and A and B be nonempty subsets of U. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a F-P-a-contraction. Then:*

$$\forall \epsilon > 0, \ F_\epsilon^z(A, B) \neq \varnothing.$$

Proof: Let $\epsilon > 0$ and $u \in A \cup B$.

$$\begin{aligned} \wedge\{t > 0 : N(T^n u - T^{n+1}u, t) \geq \alpha\} &= \wedge\{t > 0 : N(T(T^{n-1}u) - T(T^n u), t) \geq \alpha\} \\ &\leq a \wedge \{t > 0 : N(T^{n-1}u - T^n u, t) \geq \alpha\} \\ &\ \ \vdots \\ &\leq a^n \wedge \{t > 0 : N(u - Tu, t) \geq \alpha\}. \end{aligned}$$

But $a \in (0, 1)$ Therefore

$$Lim_{n\to\infty} \wedge \{t > 0 : N(T^n u - T^{n+1}x, t) \geq \alpha\} = 0, \ \forall u \in A \cup B.$$

Now by Lemma 3.1 it follows that $F_\epsilon^z(A, B) \neq \varnothing, \forall \epsilon > 0$. ■

In 1968, Kannan (see [14],[15]) proved a fixed point theorem for operators which need not be continuous. We apply it to fuzzy normed space for fuzzy approximate best proximity pair.

Definition 3.4. Let (U, N) be a fuzzy normed space and A and B be nonempty subsets of U. A mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a F-P-Kannan operator if there exists $a \in (0, \frac{1}{2})$ such that

$$\begin{aligned} \wedge\{t > 0 : N(Tu - Tv, t) \geq \alpha\} &\leq a[\wedge\{t > 0 : N(u - Tu, t) \geq \alpha\} \\ &+ \wedge\{t > 0 : N(v - Tv, t) \geq \alpha\}], \end{aligned}$$

for all $u, v \in A \cup B$.

Theorem 3.5. *Let (U, N) be a fuzzy normed space and A and B be nonempty subsets of U. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a F-P-Kannan operator. Then:*

$$\forall \epsilon > 0, \ F_\epsilon^z(A, B) \neq \emptyset.$$

Proof: Let $\epsilon > 0$ and $u \in A \cup B$.

$$\begin{aligned}\wedge\{t>0:N(T^n u - T^{n+1}u,t)\geq\alpha\} &= \wedge\{t>0:N(T(T^{n-1}u)-T(T^n u),t)\geq\alpha\}\\ &\leq a[\wedge\{t>0:N(T^{n-1}u-T(T^{n-1}u),t)\geq\alpha\}\\ &+ \wedge\{t>0:N(T^n u-T(T^n u),t)\geq\alpha\}]\\ &= a\wedge\{t>0:N(T^{n-1}u-T(T^{n-1}u),t)\geq\alpha\}\\ &+ a\wedge\{t>0:N(T^n u-T(T^n u),t)\geq\alpha\}.\end{aligned}$$

Therefore

$$(1-a)\wedge\{t>0:N(T^n u-T^{n+1}u,t)\geq\alpha\}\leq a\wedge\{t>0:N(T^{n-1}u-T^n u,t)\geq\alpha\}$$

Then

$$\begin{aligned}\wedge\{t>0:N(T^n u-T^{n+1}u,t)\geq\alpha\} &\leq \frac{a}{1-a}\wedge\{t>0:N(T^{n-1}u-T^n u,t)\geq\alpha\}\\ &\vdots\\ &\leq (\frac{a}{1-a})^n\wedge\{t>0:N(u-Tu,t)\geq\alpha\}\end{aligned}$$

But $a\in(0,\frac{1}{2})$ hence $\frac{a}{1-a}\in(0,1)$. Therefore

$$Lim_{n\to\infty}\wedge\{t>0:N(T^n u-T^{n+1}x,t)\geq\alpha\}=0,\ \forall u\in A\cup B.$$

Now by Lemma 3.1 it follows that $F_\epsilon^z(A,B)\neq\varnothing,\forall\epsilon>0$. ■

In 1972, Chatterjea (see [16]) considered another which again does not impose the continuity of the operator. We apply it to fuzzy normed space for fuzzy approximate best proximity pair.

Definition 3.5. Let (U,N) be a fuzzy normed space and A and B be nonempty subsets of U. A mapping $T:A\cup B\to A\cup B$ satisfying $T(A)\subseteq B$, $T(B)\subseteq A$ is a F-P-Chatterjea operator if there exists $a\in(0,\frac{1}{2})$ such that

$$\begin{aligned}\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\} &\leq a[\wedge\{t>0:N(u-Tv,t)\geq\alpha\}\\ &+ \wedge\{t>0:N(v-Tu,t)\geq\alpha\}],\end{aligned}$$

for all $u,v\in A\cup B$.

Theorem 3.6. *Let (U,N) be a fuzzy normed space and A and B be nonempty subsets of U. Suppose that the mapping $T:A\cup B\to A\cup B$ satisfying $T(A)\subseteq B$, $T(B)\subseteq A$ is a F-P-Chatterjea operator. Then:*

$$\forall\epsilon>0,\ F_\epsilon^z(A,B)\neq\emptyset.$$

Proof: Let $\epsilon > 0$ and $u \in A \cup B$.

$$\begin{aligned}\wedge\{t>0:N(T^n u-T^{n+1}u,t)\geq\alpha\} &= \wedge\{t>0:N(T(T^{n-1}u)-T(T^n u),t)\geq\alpha\}\\ &\leq a[\wedge\{t>0:N(T^{n-1}u-T(T^n u),t)\geq\alpha\}\\ &+ \wedge\{t>0:N(T^n u-T(T^{n-1}u),t)\geq\alpha\}]\\ &= a\wedge\{t>0:N(T^{n-1}u-T^{n+1}u,t)\geq\alpha\}\\ &+ a\wedge\{t>0:N(T^n u-T^n u,t)\geq\alpha\}\\ &= a[\wedge\{t>0:N(T^{n-1}u-T^{n+1}u,t)\geq\alpha\}.\end{aligned}$$

On the other hand

$$\begin{aligned}\wedge\{t>0:N(T^{n-1}u-T^{n+1}u,t)\geq\alpha\} &\leq \wedge\{t>0:N(T^{n-1}u-T^{n}u,t)\geq\alpha\}\\ &+ \wedge\{t>0:N(T^{n}u-T^{n+1}u,t)\geq\alpha\}.\end{aligned}$$

Then

$$(1-a)\wedge\{t>0:N(T^{n}u-T^{n+1}u,t)\geq\alpha\}\leq a\wedge\{t>0:N(T^{n-1}u-T^{n}u,t)\geq\alpha\}$$

Hence

$$\begin{aligned}\wedge\{t>0:N(T^{n}u-T^{n+1}u,t)\geq\alpha\} &\leq \frac{a}{1-a}\wedge\{t>0:N(T^{n-1}u-T^{n}u,t)\geq\alpha\}\\ &\vdots\\ &\leq (\frac{a}{1-a})^{n}\wedge\{t>0:N(u-Tu,t)\geq\alpha\}\end{aligned}$$

But $a\in(0,\frac{1}{2})$ hence $\frac{a}{1-a}\in(0,1)$. Therefore

$$Lim_{n\to\infty}\wedge\{t>0:N(T^{n}u-T^{n+1}x,t)\geq\alpha\}=0,\ \forall u\in A\cup B.$$

Now by Lemma 3.1 it follows that $F^{z}_{\epsilon}(A,B)\neq\varnothing,\forall\epsilon>0$. ■

We, by combining the three independent contraction conditions above obtain another fuzzy approximate best proximity pair result for operators which satisfy the following.

Definition 3.6. Let (U,N) be a fuzzy normed space and A and B be nonempty subsets of U. A mapping $T:A\cup B\to A\cup B$ satisfying $T(A)\subseteq B$, $T(B)\subseteq A$ is a F-P-Zamfirescu operator if there exists $\alpha_1,\beta,\gamma\in R$, $\alpha_1\in[0,1[,\beta\in[0,\frac{1}{2}[,\gamma\in[0,\frac{1}{2}[$ such that for all $x,y\in A\cup B$ at least one of the following is true:

$$(F_{Z1}):\ \wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq\alpha_1\wedge\{t>0:N(u-v,t)\geq\alpha\};$$

$$\begin{aligned}(F_{Z2}):\ \wedge\{t>0:N(Tu-Tv,t)\geq\alpha\} &\leq \beta[\wedge\{t>0:N(u-Tu,t)\geq\alpha\}\\ &+ \wedge\{t>0:N(v-Tv,t)\geq\alpha\}],\end{aligned}$$

$$\begin{aligned}(F_{Z3}):\ \wedge\{t>0:N(Tu-Tv,t)\geq\alpha\} &\leq \gamma[\wedge\{t>0:N(u-Tv,t)\geq\alpha\}\\ &+ \wedge\{t>0:N(v-Tu,t)\geq\alpha\}].\end{aligned}$$

Theorem 3.7. *Let (U,N) be a fuzzy normed space and A and B be nonempty subsets of U. A mapping $T:A\cup B\to A\cup B$ satisfying $T(A)\subseteq B$, $T(B)\subseteq A$ is a F-P-Zamfirescu operator. Then:*

$$\forall\epsilon>0,\ F^{z}_{\epsilon}(A,B)\neq\emptyset.$$

Proof: Let $u,v\in A\cup B$. Supposing $F_{Z2})$ holds, we have that:

$$\begin{aligned}\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\} &\leq \beta[\wedge\{t>0:N(u-Tu,t)\geq\alpha\}+\wedge\{t>0:N(v-Tv,t)\geq\alpha\}]\\ &\leq \beta\wedge\{t>0:N(u-Tu,t)\geq\alpha\}+\beta[\wedge\{t>0:N(v-u,t)\geq\alpha\}\\ &+ \wedge\{t>0:N(u-Tu,t)\geq\alpha\}+\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}]\\ &= 2\beta\wedge\{t>0:N(u-Tu,t)\geq\alpha\}+\beta\wedge\{t>0:N(u-v,t)\geq\alpha\}\\ &+ \beta\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}.\end{aligned}$$

Thus

$$\begin{aligned}\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\} &\leq \frac{2\beta}{1-\beta}\wedge\{t>0:N(u-Tu,t)\geq\alpha\}\\ &+ \frac{\beta}{1-\beta}\wedge\{t>0:N(u-v,t)\geq\alpha\}. \qquad (1)\end{aligned}$$

Supposing F_{Z3}) holds, we have that:

$$\begin{aligned}\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\} &\leq \gamma[\wedge\{t>0:N(u-Tv,t)\geq\alpha\}\\ &+ \wedge\{t>0:N(v-Tu,t)\geq\alpha\}]\\ &\leq \gamma[\wedge\{t>0:N(u-v,t)\geq\alpha\}+\wedge\{t>0:N(v-Tv,t)\geq\alpha\}]\\ &+ \gamma[\wedge\{t>0:N(v-Tv,t)\geq\alpha\}]+\gamma[\wedge\{t>0:N(v-Tv,t)\geq\alpha\}\\ &+ \wedge\{t>0:N(Tv-Tu,t)\geq\alpha\}]=\gamma\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\\ &+ 2\gamma\wedge\{t>0:N(v-Tv,t)\geq\alpha\}+\gamma\wedge\{t>0:N(u-v,t)\geq\alpha\}.\end{aligned}$$

Thus

$$\begin{aligned}\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\} &\leq \frac{2\gamma}{1-\gamma}\wedge\{t>0:N(v-Tv,t)\geq\alpha\}\\ &+ \frac{\gamma}{1-\gamma}\wedge\{t>0:N(u-v,t)\geq\alpha\}. \qquad (2)\end{aligned}$$

Similarly:

$$\begin{aligned}\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\} &\leq \gamma[\wedge\{t>0:N(u-Tv,t)\geq\alpha\}\\ &+ \wedge\{t>0:N(v-Tu,t)\geq\alpha\}]\\ &\leq \gamma[\wedge\{t>0:N(u-Tu,t)\geq\alpha\}+\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}]\\ &+ \gamma[\wedge\{t>0:N(v-u,t)\geq\alpha\}+\gamma[\wedge\{t>0:N(u-Tu,t)\geq\alpha\}]\\ &= \gamma\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}+2\gamma\wedge\{t>0:N(u-Tu,t)\geq\alpha\}\\ &+ \gamma\wedge\{t>0:N(u-v,t)\geq\alpha\}.\end{aligned}$$

Then

$$\begin{aligned}\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\} &\leq \frac{2\gamma}{1-\gamma}\wedge\{t>0:N(u-Tu,t)\geq\alpha\}\\ &+ \frac{\gamma}{1-\gamma}\wedge\{t>0:N(u-v,t)\geq\alpha\}. \qquad (3)\end{aligned}$$

Now looking at F_{Z1}), (1), (2), (3) we can denote:

$$\eta = max\{\alpha_1, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\},$$

and it is easy to see that $\eta\in[0,1[$.

For T satisfying at least one of the conditions F_{Z1}),F_{Z2}), F_{Z3}) we have that

$$\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq 2\eta\wedge\{t>0:N(u-Tu,t)\geq\alpha\}+\eta\wedge\{t>0:N(u-v,t)\geq\alpha\} \qquad (4)$$

and

$$\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq 2\eta\wedge\{t>0:N(v-Tv,t)\geq\alpha\}+\eta\wedge\{t>0:N(u-v,t)\geq\alpha\} \qquad (5)$$

hold.

Using these conditions implied by $F_{Z1}) -\ F_{Z3})$ and taking $u \in A \cup B$, we have:

$$\begin{array}{rcl}\wedge\{t>0:N(T^nu-T^{n+1}u,t)\geq\alpha\} & = & \wedge\{t>0:N(T(T^{n-1}u)-T(T^nu),t)\geq\alpha\}\\ & \leq^{(4)} & 2\eta[\wedge\{t>0:N(T^{n-1}u-T(T^{n-1}u),t)\geq\alpha\}\\ & + & \eta\wedge\{t>0:N(T^{n-1}u-T(T^nu),t)\geq\alpha\}\\ & = & 3\eta\wedge\{t>0:N(T^{n-1}u-T^nu,t)\geq\alpha\}.\end{array}$$

Then

$$\wedge\{t>0:N(T^nu-T^{n+1}u,t)\geq\alpha\} \quad \leq\cdots\leq \quad (3\eta)^3\wedge\{t>0:N(u-Tu),t)\geq\alpha\}.$$

Therefore

$$Lim_{n\to\infty}\wedge\{t>0:N(T^nu-T^{n+1}x,t)\geq\alpha\}=0,\ \forall u\in A\cup B.$$

Now by Lemma 3.1 it follows that $F^z_\epsilon(T)\neq\varnothing, \forall\epsilon>0$. ■

Now, we consider the contraction condition given in 2004 by V. Berinde, who also formulated a corresponding fixed point theorem, see [15], for example.

Definition 3.7. Let (U, N) be a fuzzy normed space and A and B be nonempty subsets of U. A mapping $T : A\cup B \to A\cup B$ satisfying $T(A)\subseteq B$, $T(B)\subseteq A$ is a F-P-weak contraction operator if there exists $\alpha_1 \in]0, 1[$ and $L \geq 0$ such that

$$\begin{array}{rcl}\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\} & \leq & \alpha_1\wedge\{t>0:N(u-v,t)\geq\alpha\}\\ & + & L\wedge\{t>0:N(v-Tu,t)\geq\alpha\}\end{array}$$

for all $u, v \in A\cup B$.

Theorem 3.8. *Let (U, N) be a fuzzy normed space and A and B be nonempty subsets of U. A mapping $T : A\cup B\to A\cup B$ satisfying $T(A)\subseteq B$, $T(B)\subseteq A$ is a F-P-weak contraction. Then:*

$$\forall\epsilon>0,\ F^z_\epsilon(A,B)\neq\emptyset.$$

Proof: Let $u \in A\cup B$.

$$\begin{array}{rcl}\wedge\{t>0:N(T^nu-T^{n+1}u,t)\geq\alpha\} & = & \wedge\{t>0:N(T(T^{n-1}u)-T(T^nu),t)\geq\alpha\}\\ & \leq & \alpha_1\wedge\{t>0:N(T^{n-1}u-T^nu),t)\geq\alpha\}\\ & + & L\wedge\{t>0:N(T^nu-T^nu,t)\geq\alpha\}\\ & = & \alpha_1\wedge\{t>0:N(T^{n-1}u-T^nu,t)\geq\alpha\}\\ & \leq\cdots\leq & \alpha_1^n\wedge\{t>0:N(u-Tu,t)\geq\alpha\}.\end{array}$$

But $\alpha_1 \in [0, 1[$. Therefore

$$Lim_{n\to\infty}\wedge\{t>0:N(T^nu-T^{n+1}x,t)\geq\alpha\}=0,\ \forall u\in U.$$

Now by Lemma 3.1 it follows that $F^z_\epsilon(A,B)\neq\varnothing, \forall\epsilon>0$. ■

In 1974, Ciric [17] obtained a contraction condition for which the map satisfying it is still a Picard operator. We apply it to fuzzy normed space for fuzzy approximate best proximity pair.

Definition 3.8. Let (U, N) be a fuzzy normed space and A and B be nonempty subsets of U. A mapping $T : A\cup B\to A\cup B$ satisfying $T(A)\subseteq B$, $T(B)\subseteq A$ is a F-P-quasi contraction if there exists $h \in]0, 1[$ such that

$$\begin{array}{rcl}\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\} & \leq & h\cdot max\{\wedge\{t>0:N(u-v,t)\geq\alpha\},\\ & & \wedge\{t>0:N(u-Tu,t)\geq\alpha\},\wedge\{t>0:N(v-Tv,t)\geq\alpha\},\\ & & \wedge\{t>0:N(u-Tv,t)\geq\alpha\},\wedge\{t>0:N(v-Tu,t)\geq\alpha\}\}\end{array}$$

for all $u, v \in A\cup B$.

Corollary 3.9. *Let (U,N) be a fuzzy normed space and A and B be nonempty subsets of U. A mapping $T : A\cup B \to A\cup B$ satisfying $T(A)\subseteq B$, $T(B)\subseteq A$ is a F-P-quasi contraction with $0 < h < \frac{1}{2}$. Then:*

$$\forall\epsilon > 0,\ F^z_\epsilon(A,B) \neq \emptyset.$$

Proof: By Proposition 3 of [18] any P-quasi contraction with $0 < h < \frac{1}{2}$ is a weak contraction. Therfore by Theorem 3.8, $F^z_\epsilon(A,B) \neq \emptyset, \forall\epsilon > 0$.

4 Diameter Fuzzy Approximate Best Proximity Pair for Various Types of Operators

For the same operators we have studied in the previous section, we will formulate and prove useing Lemma 3.2, in order to obtain results for diameter approximate best proximity pair.

Theorem 4.1. *Let (U,N) be a fuzzy normed space and A and B be nonempty subsets of U. Suppose that the mapping $T : A\cup B \to A\cup B$ satisfying $T(A)\subseteq B$, $T(B)\subseteq A$ is a F-P-a-contraction, and $\epsilon > 0$. Then for some $\alpha\in(0,1)$:*

$$diam(F^z_\epsilon(A,B)) \leq \frac{2\epsilon}{1-a} + \frac{2(\wedge\{t>0 : N(A-B,t)\geq\alpha\}}{1-a}.$$

Proof: Let $\epsilon_1,\epsilon_2 > 0$ and $u,v\in F^z_\epsilon(A,B)$. Then:

$$\wedge\{t>0 : N(u-Tu,t)\geq\alpha\} \leq \wedge\{t>0 : N(A-B,t)\geq\alpha\} + \epsilon_1$$

and

$$\wedge\{t>0 : N(v-Tv,t)\geq\alpha\} \leq \wedge\{t>0 : N(A-B,t)\geq\alpha\} + \epsilon_2.$$

We can write:

$$\wedge\{t>0 : N(u-v,t)\geq\alpha\} \leq \wedge\{t>0 : N(u-Tu,t)\geq\alpha\} + \wedge\{t>0 : N(Tu-Tv,t)\geq\alpha\}$$

$$+\wedge\{t>0 : N(v-Tv,t)\geq\alpha\} \leq a\wedge\{t>0 : N(u-v,t)\geq\alpha\}+2(\wedge\{t>0 : N(A-B,t)\geq\alpha\})+\epsilon_1+\epsilon_2$$

Put $\epsilon = Max\{\epsilon_1,\epsilon_2\}$. Then,

$$\wedge\{t>0 : N(u-v,t)\geq\alpha\} \leq \frac{2\epsilon}{1-a} + \frac{2\wedge\{t>0 : N(A-B,t)\geq\alpha\}}{1-a}.$$

Hence

$$diam(F^z_\epsilon(A,B)) \leq \frac{2\epsilon}{1-a} + \frac{2\wedge\{t>0 : N(A-B,t)\geq\alpha\}}{1-a}$$

for some $\alpha\in(0,1)$. ■

Theorem 4.2. *Let (U,N) be a fuzzy normed space and A and B be nonempty subsets of U. Suppose that the mapping $T : A\cup B \to A\cup B$ satisfying $T(A)\subseteq B$, $T(B)\subseteq A$ is a F-P-Kannan operator and $\epsilon > 0$. Then for some $\alpha\in(0,1)$:*

$$diam(F^z_\epsilon(A,B)) \leq 2\epsilon(1+a) + 2a\wedge\{t>0 : N(A-B,t)\geq\alpha\}.$$

Proof: Let $\epsilon > 0$. Condition 1) in Lemma 3.2, is satisfied, as one can see in the proof of Theorem 3.5, we only verify that condition 2) in Lemma 3.2, holds.

Let $\theta > 0$ and $u,v\in F^z_\epsilon(A,B))$ and assume that

$$\wedge\{t>0 : N(u-v,t)\geq\alpha\} - \wedge\{t>0 : N(Tu-Tv,t)\geq\alpha\} \leq \theta.$$

Then:

$$\begin{aligned} \wedge\{t>0 : N(u-v,t)\geq\alpha\} &\leq a[\wedge\{t>0 : N(u-Tu,t)\geq\alpha\} \\ &+ \wedge\{t>0 : N(v-Tv,t)\geq\alpha\}]+\theta. \end{aligned}$$

As $u, v \in F^z_\epsilon(A,B))$, we know that

$$\wedge\{t>0 : N(u-Tu,t)\geq\alpha\}\leq\wedge\{t>0 : N(A-B,t)\geq\alpha\}+\epsilon_1$$

and

$$\wedge\{t>0 : N(v-Tv,t)\geq\alpha\}\leq\wedge\{t>0 : N(A-B,t)\geq\alpha\}+\epsilon_2.$$

Put $\epsilon=\{\epsilon_1,\epsilon_2\}$. Then,

$$\wedge\{t>0 : N(u-v,t)\geq\alpha\}\leq 2a\wedge\{t>0 : N(A-B,t)\geq\alpha\}+2a\epsilon+\theta.$$

So for every $\theta>0$ there exists $\phi(\theta)=\theta+2a(\wedge\{t>0 : N(A-B,t)\geq\alpha\}+\epsilon)>0$ such that

$$\wedge\{t>0 : N(u-v,t)\geq\alpha\}-\wedge\{t>0 : N(Tu-Tv,t)\geq\alpha\}\leq\theta$$

$$\Rightarrow\wedge\{t>0 : N(u-v,t)\geq\alpha\}\leq\phi(\theta).$$

Now by Lemma 3.2, it follows that

$$diam(F^z_\epsilon(A,B))\leq\phi(2\epsilon),\forall\epsilon>0,$$

which means exactly that

$$diam(F^z_\epsilon(A,B))\leq 2\epsilon(1+\alpha)+2\alpha\wedge\{t>0 : N(A-B,t)\geq\alpha\},\forall\epsilon>0. \blacksquare$$

Theorem 4.3. *Let (U,N) be a fuzzy normed space and A and B be nonempty subsets of U. Suppose that the mapping $T : A\cup B\to A\cup B$ satisfying $T(A)\subseteq B$, $T(B)\subseteq A$ is a F-P-Chatterjea operator and $\epsilon>0$. Then for some $\alpha\in(0,1)$:*

$$diam(F^z_\epsilon(A,B))\leq\frac{2\epsilon(1+a)+2a\wedge\{t>0 : N(A-B,t)\geq\alpha\}}{1-2a}.$$

Proof: Let $\epsilon>0$. We will only verify that condition 2) in Lemma 3.2 holds.

Let $\theta>0$ and $u, v\in F^z_\epsilon(A,B))$ and assume that

$$\wedge\{t>0 : N(u-v,t)\geq\alpha\}-\wedge\{t>0 : N(Tu-Tv,t)\geq\alpha\}\leq\theta.$$

Then:

$$\begin{aligned} \wedge\{t>0 : N(u-v,t)\geq\alpha\} &\leq a[\wedge\{t>0 : N(u-Tv,t)\geq\alpha\} \\ &+ \wedge\{t>0 : N(v-Tu,t)\geq\alpha\}]+\theta \\ &\leq a\wedge\{t>0 : N(u-Tv,t)\geq\alpha\} \\ &+ a\wedge\{t>0 : N(v-Tu,t)\geq\alpha\}+\theta \\ &\leq a[\wedge\{t>0 : N(u-v,t)\geq\alpha\} \\ &+ \wedge\{t>0 : N(v-Tv,t)\geq\alpha\}] \\ &+ a[\wedge\{t>0 : N(v-u,t)\geq\alpha\} \\ &+ \wedge\{t>0 : N(u-Tu,t)\geq\alpha\}]+\theta. \end{aligned}$$

As $u, v\in F^z_\epsilon(A,B))$, we know that

$$\wedge\{t>0 : N(u-Tu,t)\geq\alpha\}\leq\wedge\{t>0 : N(A-B,t)\geq\alpha\}+\epsilon_1,$$

$$\wedge\{t>0:N(v-Tv,t)\geq\alpha\}\leq\wedge\{t>0:N(A-B,t)\geq\alpha\}+\epsilon_2.$$

Put $\epsilon=\{\epsilon_1,\epsilon_2\}$. Then,

$$\begin{aligned}\wedge\{t>0:N(u-v,t)\geq\alpha\} &\leq 2a\wedge\{t>0:N(u-v,t)\geq\alpha\}\\ &+ 2a\wedge\{t>0:N(A-B,t)\geq\alpha\}+2a\epsilon+\theta.\end{aligned}$$

Then

$$(1-2a)\wedge\{t>0:N(u-v,t)\geq\alpha\}\leq 2a\wedge\{t>0:N(A-B,t)\geq\alpha\}+2a\epsilon+\theta.$$

So,

$$\wedge\{t>0:N(u-v,t)\geq\alpha\}\leq\frac{2a(\wedge\{t>0:N(A-B,t)\geq\alpha\}+\epsilon)+\theta}{(1-2a)}.$$

$$\wedge\{t>0:N(u-v,t)\geq\alpha\}-\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq\theta.$$

$$\Rightarrow\wedge\{t>0:N(u-v,t)\geq\alpha\}\leq\phi(\theta).$$

Now by Lemma 3.2, it follows that

$$diam(F_\epsilon^z(A,B))\leq\phi(2\epsilon),\forall\epsilon>0,$$

which means exactly that

$$diam(F_\epsilon^z(A,B))\leq\frac{2\epsilon(1+a)+2a\wedge\{t>0:N(A-B,t)\geq\alpha\}}{1-2a},\forall\epsilon>0. \blacksquare$$

Theorem 4.4. *Let (U,N) be a fuzzy normed space and A and B be nonempty subsets of U. Suppose that the mapping $T:A\cup B\to A\cup B$ satisfying $T(A)\subseteq B$, $T(B)\subseteq A$ is a F-P-Zamfirescu operator and $\epsilon>0$. Then for some $\alpha\in(0,1)$:*

$$diam(F_\epsilon^z(A,B))\leq\frac{2\epsilon(1+\eta)+2\eta\wedge\{t>0:N(A-B,t)\geq\alpha\}}{1-\eta}.$$

where $\eta=max\{\alpha_1,\frac{\beta}{1-\beta},\frac{\gamma}{1-\gamma}\}$, and α_1,β,γ as in Definition 3.6.

Proof: In the proof of Theorem 3.7, we have already shown that if T satisfies at least one of the conditions F_{Z1}),F_{Z2}), F_{Z3}) from Definition 3.6, then

$$\begin{aligned}\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\} &\leq 2\eta\wedge\{t>0:N(u-Tu,t)\geq\alpha\}\\ &+ \eta\wedge\{t>0:N(u-v,t)\geq\alpha\}\end{aligned}$$

and

$$\begin{aligned}\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\} &\leq 2\eta\wedge\{t>0:N(v-Tv,t)\geq\alpha\}\\ &+ \eta\wedge\{t>0:N(u-v,t)\geq\alpha\}\end{aligned}$$

hold.

Let $\epsilon>0$. We will only verify that condition 2) in Lemma 3.2 is satisfied, as 1) holds, see the proof of Theorem 3.7.
Let $\theta>0$ and $x,y\in F_\epsilon^z(A,B))$ and assume that

$$\wedge\{t>0:N(u-v,t)\geq\alpha\}-\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq\theta.$$

Then

$$\begin{aligned}\wedge\{t>0:N(u-v,t)\geq\alpha\} &\leq \wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}+\theta\\ &\leq 2\eta\wedge\{t>0:N(u-Tu,t)\geq\alpha\}\\ &+ \eta\wedge\{t>0:N(u-v,t)\geq\alpha\}+\theta\end{aligned}$$

$$\Rightarrow (1-\eta)\wedge\{t>0:N(u-v,t)\geq\alpha\}\leq 2\eta\wedge\{t>0:N(A-B,t)\geq\alpha\}+2\eta\epsilon+\theta$$

$$\wedge\{t>0:N(u-v,t)\geq\alpha\}\leq\frac{2\eta(\wedge\{t>0:N(A-B,t)\geq\alpha\}+\epsilon)+\theta}{1-\eta}.$$

So for every $\theta>0$ there exists $\phi(\theta)=\frac{2\eta(\wedge\{t>0:N(A-B,t)\geq\alpha\}+\epsilon)+\theta}{1-\eta}>0$ such that

$$\wedge\{t>0:N(u-v,t)\geq\alpha\}-\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq\theta.$$

$$\Rightarrow\wedge\{t>0:N(u-v,t)\geq\alpha\}\leq\phi(\theta).$$

Now by Lemma 3.2, it follows that

$$diam(F_\epsilon^z(A,B))\leq\phi(2\epsilon),\forall\epsilon>0,$$

which means exactly that

$$diam(F_\epsilon^z(A,B))\leq\frac{2\epsilon(1+\eta)+2\eta\wedge\{t>0:N(A-B,t)\geq\alpha\}}{1-\eta},\forall\epsilon>0.\ \blacksquare$$

Theorem 4.5. *Let (U,N) be a fuzzy normed space and A and B be nonempty subsets of U. Suppose that the mapping $T:A\cup B\to A\cup B$ satisfying $T(A)\subseteq B$, $T(B)\subseteq A$ is a P-weak contraction with $\alpha_1+L<1$, and $\epsilon>0$. Then for some $\alpha\in(0,1)$:*

$$diam(F_\epsilon^z(A,B))\leq\frac{(L+2)\epsilon+L\wedge\{t>0:N(A-B,t)\geq\alpha\}}{1-\alpha_1-L}.$$

Proof: Let $\epsilon>0$. We will only verify that condition 2) in Lemma 3.2 holds.

Let $\theta>0$ and $x,y\in F_\epsilon^z(A,B))$ and assume that

$$\wedge\{t>0:N(u-v,t)\geq\alpha\}-\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq\theta.$$

Then

$$\begin{aligned}\wedge\{t>0:N(u-v,t)\geq\alpha\} &\leq \wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}+\theta\\ &\leq \alpha_1\wedge\{t>0:N(u-v,t)\geq\alpha\}\\ &+ L\wedge\{t>0:N(v-Tu,t)\geq\alpha\}+\theta\\ &\leq \alpha_1\wedge\{t>0:N(u-v,t)\geq\alpha\}\\ &+ L\wedge\{t>0:N(u-v,t)\geq\alpha\}\\ &+ L\wedge\{t>0:N(u-Tu,t)\geq\alpha\}+\theta\\ &\leq (\alpha_1+L)\wedge\{t>0:N(u-v,t)\geq\alpha\}\\ &+ L(\wedge\{t>0:N(A-B,t)\geq\alpha\}+\epsilon)+\theta.\end{aligned}$$

Therfore

$$\wedge\{t>0:N(u-v,t)\geq\alpha\}\leq\frac{L(\wedge\{t>0:N(A-B,t)\geq\alpha\}+\epsilon)+\theta}{1-\alpha_1-L}.$$

So for every $\theta>0$ there exists $\phi(\theta)=\frac{L(\wedge\{t>0:N(A-B,t)\geq\alpha\}+\epsilon)+\theta}{1-\alpha_1-L}>0$ such that

$$\wedge\{t>0:N(u-v,t)\geq\alpha\}-\wedge\{t>0:N(Tu-Tv,t)\geq\alpha\}\leq\theta.$$

$$\Rightarrow \wedge\{t > 0 : N(u - v, t) \geq \alpha\} \leq \phi(\theta).$$

Now by Lemma 3.2, it follows that

$$diam(F_\epsilon^z(A, B)) \leq \phi(2\epsilon), \forall \epsilon > 0,$$

which means exactly that

$$diam(F_\epsilon^z(A, B)) \leq \frac{(L + 2)\epsilon + L \wedge \{t > 0 : N(A - B, t) \geq \alpha\}}{1 - \alpha_1 - L}, \forall \epsilon > 0. \blacksquare$$

5 Conclusions

Nowadays, fixed point theory play an important role in different areas of mathematics and its applications, particularly in physics, differential equation and dynamic programming. We think that this paper could be of interest to the researchers working in the field fuzzy functional analysis in particular, fuzzy approximate best proximity pair theory are used. We proved results about approximate best proximity pair and diameter approximate best proximity pair on fuzzy normed spaces, starting from a result presented in [12], but the study may go further to other classes of operators, which will be the subject of future papers.

Acknowledgement

The authors are extremely grateful to the referees for their helpful suggestions for the improvement of the paper.

Competing Interests

The author declares that no competing interests exist.

References

[1] Zadeh LA. Fuzzy sets. Inform. Control. 1965;8:338-353.

[2] Katsaras AK. Fuzzy topological vector spaces II. Fuzzy Sets and Systems. 1984;12:143154.

[3] Felbin C. Finite dimensional fuzzy normed linear spaces. Fuzzy Sets and Systems. 1992;48:239-248.

[4] Kaleva O, Seikkala S. On fuzzy metric spaces. Fuzzy Sets and Systems. 1984;12:215-229.

[5] Cheng SC, Mordeson JN. Fuzzy linear operator and fuzzy normed linear spaces. Bull. Calcutta Math. Soc. 1994;86:429-436.

[6] Kramosil I, Michalek J. Fuzzy metric and statistical metric spaces. Kybernetica. 1975;11:326-334.

[7] Chitra A, Mordeson PV. Fuzzy linear operators and fuzzy normed linear spaces. Bull. Calcutta Math. Soc. 1969;74:660-665.

[8] Bag T, Samanta SK. Finite dimensional fuzzy normed linear spaces. J. Fuzzy Math. 2003;11:687-705.

[9] Bag T, Samanta SK. Fuzzy bounded linear operators. Fuzzy Sets and Systems. 2005;151:513-547.

[10] Bag T, Samanta SK. Some fixed point theorems in fuzzy normed linear spaces. Information Sciences. 2007;177:3271-3289.

[11] Mohsenalhosseini SAM, Mazaheri H, Dehghan MA. Approximate fixed point in fuzzy normed spaces for nonlinear maps. Iranin Journal of Fuzzy Systems. 2013;10:135-142.

[12] Mohsenalhosseini SAM. Approximate best proximity pairs in metric space for contraction maps. Advances in Fixed Point Theory. 2014;4:310-324.

[13] Mohsenalhosseini SAM. Approximate fixed point theorems in fuzzy norm spaces for an operator. Advances in Fuzzy Systems, Systems; 2013. Article ID 613604, 8 pages. Napoca; 2001.

[14] Kannan R. Some results on fixed points. Bull. Calcutta Math. Soc. 1968;10:71-76.

[15] Berinde V. On the approximation of fixed points of weak contractive mappings. Math. 2003;19:7-22.

[16] Chatterjea SK. Fixed-point theorems. C.R. Acad. Bulgare Sci. 1972;25:727-730.

[17] Lj B. Ciric. A generalization of Banachs contraction principle. Proc. Am. Math. Soc. 1974;45:267-273.

[18] Berinde V. Approximating fixed points of weak contractions using the Picard iteration. Nonlinear Analysis Forum. 2004;9.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sciencedomain.org/review-history/27082