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Fuzzy Group Derived From Filtered Ring and Its Properties

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript

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Abstract

In this paper we show if R is a filtered ring, then we can define fuzzy subgroup. Then we show that it has abelian fuzzy subset some of fuzzy- group and fuzzy set properties. Also we prove some properties for strongly filtered ring.

Keywords: Filtered ring; fuzzy set; fuzzy group; fuzzy abelian subset.

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1 Introduction

The pioneering work of Zadeh on fuzzy subsets of a set in [1] and Rosenfeld on fuzzy subgroups of a group in [2] led to the fuzzification of algebraic structures. In this chapter, we begin the study of fuzzy subgroups of a group. We assume that the least upper bound of the empty set is 0 and the greatest lower bound of the empty set is 1. We let X, Y and Z denote nonempty sets. We

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let $\mathbb{N}$ denote the set of positive integers, $\mathbb{Z}$ the set of all integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and ϕ the empty set. Unless otherwise stated I denotes an arbitrary nonempty index set. Many of the results of this article are taken from Yu, Mordeson and Cheng [3]. An extensive list of references of the material used in the development of [3] can be found there, for example see [4], [2], [5] and [6]. On the other hand we know that filtered ring is also a most important structure since filtered ring is a base for graded ring especially associated graded ring and completion and some result like on the Andreadakis–Johnson filtration of the automorphism group of a free group [7] on the depth of the associated graded ring of a filtration[8], [9]. So as these important structure the relation between these structure is useful for finding some new structure.

In this article we define a fuzzy group for a filtered ring, and find many new relation and structure.

2 Preliminaries

We write $\wedge$ for minimum or infimum and $\vee$ for maximum or supremum

A fuzzy subset of X is a function from X into $[0, 1]$. The set of all fuzzy subset of X is called the fuzzy power set of X and is denoted by $FP(X)$. Let $\mu \in FP(X)$. Then the set $\{\mu(x)\,|x \in X\}$ is called image of μ. It is denoted by $\mu(X)$ or $Im(\mu)$. The set $\{x\,|x \in X, \mu(x) > 0\}$, is called the support of μ, and is denoted by μ^*. In particular, μ is called a finite fuzzy subset, if μ^* is a finite set, and an infinite fuzzy subset otherwise. Let $Y \subseteq X$ and $a \in [0, 1]$. We define $a_Y \in FP(X)$ as follows:

$$a_Y(x) = \begin{cases} a \quad for \quad x \in Y \\ 0 \quad for \quad x \in X\backslash Y \end{cases}$$

Let G be a set. We define the binary operation "$\circ$" on $FP(G)$ and the unary operation "$^{-1}$" on $FP(G)$ as follows:

$$\forall \mu, \nu \in FP(G) \text{ and } \forall x \in G$$

Then we have

$$(\mu \circ \nu)(x) = \vee\{\mu(y) \wedge \nu(z)\,|y, z \in G, yz = x\}$$

and

$$\mu^{-1}(x) = \mu(x^{-1}).$$

Let $\mu \in FP(G)$. Then μ is called a fuzzy subgroup of G, if:

$$\mu(xy) \geq \mu(x) \wedge \mu(y) \qquad \forall x, y \in G$$

, and

$$\mu(x^{-1}) \geq \mu(x) \qquad \forall x \in G \ .$$

The set of all fuzzy subgroups of G denoted by $F(G)$.

Lemma 2.1. *Let $\mu \in FP(G)$. Then $\forall x \in G$,*

$$\mu(e) \geq \mu(x);$$

$$\mu(x) = \mu(x^{-1})$$

Proof. see([8], [2]). □

Let $\mu \in FP(G)$, and

$$\langle\mu\rangle = \bigcap \{\nu\,|\mu \subseteq \nu, \nu \in F(G)\}\,.$$

Then $\langle\mu\rangle$ is called the fuzzy subgroup of G generated by μ.

Theorem 2.2. *Let $\mu \in FP(G)$. Then the following assertions are equivalent:*

i)

$$\mu(xy) = \mu(yx) \qquad \forall x, y \in G$$

In this case, μ is called an Abelian fuzzy subset of G.

ii)

$$\mu(xyx^{-1}) = \mu(y) \quad \forall x, y \in G.$$

iii)

$$\mu(xyx^{-1}) \geq \mu(y) \quad \forall x, y \in G.$$

iv)

$$\mu(xyx^{-1}) \leq \mu(y) \quad \forall x, y \in G.$$

v)

$$\mu \circ \nu = \nu \circ \mu \quad \forall \nu \in FP(G).$$

Proof. see([8] ,[2]). □

Let $\mu \in F(G)$. Then μ is called a normal fuzzy subgroup of G, if it is an Abelian fuzzy subset of G. $NF(G)$ denotes the set of all normal fuzzy subgroups of G.

Remark 2.1. Let G and H be groups, and let $\mu \in F(G)$ and $\nu \in F(H)$.

A homomorphism f of G onto H is called a weak homomorphism of μ into ν, if $f(\mu) \subseteq \nu$. If f is a weak homomorphism of μ into ν, then we say that μ is weakly homomorphic to ν.

A filtered ring R is a ring together with a family $\{R_n\}_{n\geq 0}$ of subgroups of R satisfying the following conditions:

i) $R_0 = R$;

ii) $R_{n+1} \subseteq R_n$ for all $n \geq 0$;

iii) $R_n R_m \subseteq R_{n+m}$ for all $n, m \geq 0$.

Let R be a ring together with a family $\{R_n\}_{n\geq 0}$ of subgroups of R satisfying the following conditions:

i) $R_0 = R$;

ii) $R_{n+1} \subseteq R_n$ for all $n \geq 0$;

iii) $R_n R_m = R_{n+m}$ for all $n, m \geq 0$,

Then we say R has a strong filtration.

A map $f : M \to N$ is called a homomorphism of filtered modules, if: (i) f is R-module an homomorphism, and (ii) $f(M_n) \subseteq N_n$ for all $n \geq 0$

3 Fuzzy Subgroup Derived from Filtered Ring

Let R be a ring with a unit not necessary commutative ring.

Theorem 3.1. *Let R be a nontrivial filtered ring with filtration $\{R_n\}_{n\geq 0}$ such that we have $R_0 \supseteq R_1 \supseteq R_2 \ldots$, then $\mu(x) = \frac{1}{|R_n|}$ where $n = \min\{i \,|x \notin R_i\}$ is a fuzzy subgroup on $F(G)$ where $G = R$.*

Proof. We should show

$$\mu(xy) \geq \mu(x) \wedge \mu(y) \qquad \forall x, y \in G \ .$$

Now let

$$\mu(xy) = \frac{1}{|R_k|} \quad where \quad k = \min\{i \,|xy \notin R_i\}$$

$$so \quad xy \in R_{k-1}.$$

On the other hand, let

$$\mu(x) = \frac{1}{|R_n|} \quad where \quad n = \min\{i \,|x \notin R_i\}$$

and

$$\mu(y) = \frac{1}{|R_m|} \quad where \quad m = \min\{i \,|x \notin R_i\}$$

$$so \quad x \notin R_n \quad and \quad y \notin R_m.$$

Without losing the generality let $n \leq m$ and $R_m \subseteq R_n$ then $x \notin R_m$.

We have $xy \in R_{n-1}R_{m-1} \subseteq R_{m+n-2}$ since $x \in R_{n-1}$ and $y \in R_{m-1}$.
Therefore $R_k \subseteq R_{n+m-1}$ hence $\frac{1}{|R_{n+m-2}|} \leq \frac{1}{|R_{k-1}|}$.
Since $n+m-2 > n-2$ then $R_{n-2} \supseteq R_{n+m-2}$ similarly $R_{m-2} \supseteq R_{n+m-2}$.
If $R_{n+m-2} \not\subset R_{n-1}$, and $R_{n+m-2} \not\subset R_{m-1}$, then $n-1 \geq n+m-2$, and $m-1 \geq n+m-2$, so $n \geq n+m-1$, and $m \geq n+m-1$. Now we have $m \leq 1$ and $n \leq 1$ since $n < m \leq 1$, and $n, m \geq 0$, hence $R_0 \subseteq R_{n+m-2}$ then $n+m = 2$ it is a contradiction. Consequently $R_{n+m-2} \subseteq R_{n-1}$ or $R_{n+m-2} \subseteq R_{m-1}$, then $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$. So for all $x, y \in G$ we have $\mu(xy) \geq \mu(x) \wedge \mu(y)$.

Now we show

$$\mu(x^{-1}) \geq \mu(x) \qquad \forall x \in G$$

If $\mu(x^{-1}) < \mu(x)$, there exists n, m such that $x^{-1} \in R_{n-1}$, and $x \in R_{m-1}$, then $1 \in R_{n+m-2}$ so $x \in R_{n+m-3}$, and $x^{-1} \in R_{n+m-3}$, since $n, m \geq 1$ and it is contradiction. □

Lemma 3.2. *The Fuzzy Group derived from filtration as above μ is abelian fuzzy subset*

Proof. By theorem 3.1 we have

$$\mu(xy) \geq \min\{\mu(x), \mu(y)\},$$

and

$$\mu(yx) \geq \min\{\mu(x), \mu(y)\}.$$

If $\mu(xy) \neq \mu(yx)$ with out losing of generality, let $\mu(xy) > \mu(yx)$, so $\mu(xyy^{-1}) > \mu(yxy^{-1})$, then $\mu(x) > \min\{\mu(x), \mu(y)\}$, and it is contradiction. Hence $\mu(xy) = \mu(yx)$. By theorem (2.1.) μ is abelian fuzzy subset. □

Proposition 3.1. *If R is a strongly filtered ring and μ is fuzzy subgroup derived from strongly filtration. Then $\langle\mu\rangle$ is a normal fussy subgroup.*

Proof. By Theorem 3.1, and Definition 2, and Lemma 3.2. □

Proposition 3.2. *Let R and S be two filtered rings with filtration $\{R_i\}_{i\in\mathbb{Z}}$ and $\{S_i\}_{i\in\mathbb{Z}}$ with derived fuzzy group μ and ν. If $f : R \to S$ is a homomorphism of filtered module. Then f is weakly homomorphism of μ into ν, and μ is weakly homomorphic to ν.*

Proof. By Theorem 3.1 and Definition 2. □

Competing Interests

The authors declare that no competing interests exist.

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