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Lagrangian Operators with Higher Derivatives

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

A simple formal procedure makes the main properties of the ordinary lagrangian operator $\nabla_{\mathbf{q}}\mathcal{L} - \frac{d}{dt}\nabla_{\dot{\mathbf{q}}}\mathcal{L}$ extendable to some higher order differential operators defined for functions depending on the lagrangian coordinates $\mathbf{q}$ and on their derivatives of any order with respect to time. The higher order calculated expressions can provide the lagrangian components, in the classical sense of the Newton's law, for a quite general class of forces.

At the same time, the generalized equations of motions recover some of the classical alternative formulations of the Lagrangian equations.

Keywords: Lagrange's equations; higher order theories; lagrangian formalism for holonomic systems.

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1 Introduction

The starting point of our investigation is the operator

$$\mathcal{L} \longrightarrow \left(\frac{\partial \mathcal{L}}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right), \ldots, \frac{\partial \mathcal{L}}{\partial q_\ell} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\ell}\right)\right) \qquad (1.1)$$

which maps a function $\mathcal{L}$ on $\mathbb{R}^{2\ell+1}$ depending on $(q_1, \ldots, q_\ell, \dot{q}_1, \ldots, \dot{q}_\ell, t)$ onto a vector of functions on $\mathbb{R}^{3\ell+1}$ now depending on the $(3\ell+1)$–independent variables $(q_1, \ldots, q_\ell, \dot{q}_1, \ldots, \dot{q}_\ell, \ddot{q}_1, \ldots, \ddot{q}_\ell, t)$. Actually, d/dt means the total derivative operator that maps a funcion $f(y_1, \ldots, y_N, t)$ on $\mathbb{R}^{N+1}$ onto the function $\sum_{j=1}^{N} \frac{\partial f}{\partial y_j}\dot{y}_j + \frac{\partial f}{\partial t}$ on $\mathbb{R}^{2N+1}$ where $(y_1, \ldots, y_N, \dot{y}_1, \ldots, \dot{y}_N, t)$ have to be considered independent variables.

For the purpose of shortening notations, we make use of $\nabla_{\mathbf{y}}$ in order to list the derivatives with respect to $y_1, \ldots, y_N$ of a function f: $\nabla_{\mathbf{y}} f = \left(\frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_N}\right)$, so that (1.1) can be written as

$$\mathcal{L} \longrightarrow \nabla_{\mathbf{q}}\mathcal{L} - \frac{d}{dt}\left(\nabla_{\dot{\mathbf{q}}}\mathcal{L}\right)$$

It is known that a point transformation $\bar{q}_1(q_1, \ldots, q_\ell, t), \ldots\ \bar{q}_\ell(q_1, \ldots, q_\ell, t)$, even depending explicitly on t and respecting the nonsingularity condition $det\left(\frac{\partial \bar{q}_i}{\partial q_j}\right)_{i,j=1,\ldots,\ell} \neq 0$, induces on the kinetic variables the linear transformation

$$\dot{\bar{q}}_i = \sum_{j=1}^{\ell} \frac{\partial \bar{q}_i}{\partial q_j}\dot{q}_j + \frac{\partial \bar{q}_i}{\partial t}. \qquad (1.2)$$

We find it convenient to introduce the symbol $J_{\mathbf{x}}\mathbf{F}$ to denote the jacobian matrix $\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\ldots,N_2 \\ j=1,\ldots,N_1}}$ for any $\mathbf{x} = (x_1, \ldots, x_{N_1})$, $\mathbf{F} = (F_1, \ldots, F_{N_2})$. According to this notation, we can also write $\dot{\bar{\mathbf{q}}} = (J_{\mathbf{q}}\bar{\mathbf{q}})\dot{\mathbf{q}} + \frac{\partial \bar{\mathbf{q}}}{\partial t}$. If $\bar{\mathcal{L}}$ is the function $\mathcal{L}$ in the new variables, namely $\bar{\mathcal{L}}(\bar{\mathbf{q}}, \dot{\bar{\mathbf{q}}}, t) = \mathcal{L}(\mathbf{q}(\bar{\mathbf{q}}, t), \dot{\mathbf{q}}(\bar{\mathbf{q}}, \dot{\bar{\mathbf{q}}}, t), t)$, where $\mathbf{q}(\bar{\mathbf{q}}, t)$ is the inverse variable transformation inducing the kinetic transformation $\dot{\mathbf{q}}(\bar{\mathbf{q}}, \dot{\bar{\mathbf{q}}}, t) = (J_{\bar{\mathbf{q}}}\mathbf{q})\dot{\bar{\mathbf{q}}} + \frac{\partial \mathbf{q}}{\partial t}$, it makes sense to consider (1.1) in the new set of variables:

$$\bar{\mathcal{L}} \longrightarrow \nabla_{\bar{\mathbf{q}}}\bar{\mathcal{L}} - \frac{d}{dt}\left(\nabla_{\dot{\bar{\mathbf{q}}}}\bar{\mathcal{L}}\right).$$

This transformation maps $\bar{\mathcal{L}}$ onto a ℓ–vector function in the $3\ell + 1$ variables $(\bar{\mathbf{q}}, \dot{\bar{\mathbf{q}}}, \ddot{\bar{\mathbf{q}}}, t)$. Now, it is well known (or very easy to check) that the operator maps in a way that the resulting vectors are related by means of the jacobian matrix:

$$\nabla_{\bar{\mathbf{q}}}\bar{\mathcal{L}} - \frac{d}{dt}\left(\nabla_{\dot{\bar{\mathbf{q}}}}\bar{\mathcal{L}}\right) = (J_{\mathbf{q}}\bar{\mathbf{q}})^T\left(\nabla_{\mathbf{q}}\mathcal{L} - \frac{d}{dt}\left(\nabla_{\dot{\mathbf{q}}}\mathcal{L}\right)\right). \qquad (1.3)$$

where ^{T} means transposition. For each component, (1.3) writes

$$\frac{\partial \bar{\mathcal{L}}}{\partial \bar{q}_i} - \frac{d}{dt}\left(\frac{\partial \bar{\mathcal{L}}}{\partial \dot{\bar{q}}_i}\right) = \sum_{j=1}^{\ell} \frac{\partial \bar{q}_j}{\partial q_i}\left(\frac{\partial \mathcal{L}}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right)\right).$$

Thus, the effect of the change of variables $\mathbf{q} \to \bar{\mathbf{q}} = \bar{\mathbf{q}}(\mathbf{q}, t)$ is that the operator has to be multiplied by the jacobian matrix of the transformation.

Such a change rule plays a fundamental role whenever a differential expression as (1.1) is used to formulate a physical phenomenon like motion: operator (1.1) equal to the null vector is clearly the set of the Euler–Lagrange equations for a Lagrangian function $\mathcal{L} = T - V$, respectively kinetic and potential energy. From a physical point of view the rule reflects the "covariant" behaviour of the lagrangian components of the forces (including the inertial forces), namely the projections of the forces on the configuration space. In other words, the Euler–Lagrange equations transform as the components of covariant tensors.

Remark 1.1. The conversion rule (1.3) can be framed in the abstract context of a geometric point of view: operator (1.1) can be seen in terms of the jet bundles : indeed, the Euler–Lagrange equations are the components of a 1–form along the projection of the jet bundle to the configuration manifold. The covariance of the operator would be inherent in the definition; one of the first treatise in this direction is Leon and Rodrigues [1]; other references for geometric approaches are Gracia et al. [2] and Leon and Lacomba [3]. However, it is not in our mind to estabilish the framework and the developement of a geometry for our investigation, rather aiming at a physical reading of the analytical investigation.

In Section 2 we will consider a set of differential operators mapping functions depending on $\mathbf{q}$, $\dot{\mathbf{q}}$, ... at any fixed order of derivative and still performing property (1.3).

Then the question arises, about which role can play the operators within the formulation of dynamical laws for particular systems. The additional aspect, with respect to the standard theory, is evidently the presence of higher order derivatives of the lagrangian coordinates. The question is examined in Section 3.

2 Higher–order Differential Operators

The notation we will employ shows $\overset{h\cdot}{q_k}$ standing for q_k with h dots: $\overset{h\cdot}{q_k} = \overset{\overbrace{\cdot\,\cdot\,\cdot\,\cdot}^{h}}{q_k}$, that is $\dfrac{d^h q_k}{dt^h}$ (h-th derivative with respect to t); the same for $\overset{h\cdot}{\mathbf{q}} = (\overset{h\cdot}{q_1}, \dots, \overset{h\cdot}{q_\ell})$. Unless stated otherwise, the notation is valid also for $s = 0$, in the sense of the 0–derivative $\overset{0\cdot}{q_k} = q_k$.

For a fixed integer $r \geq 0$, let $\mathcal{Y}$ be a real–valued function of $\mathbf{q}$, of their time derivatives up to the order $r + 1$ and of t:

$$\mathcal{Y}(\mathbf{q}, \dot{\mathbf{q}}, \dots, \overset{r\cdot}{\mathbf{q}}, \overset{(r+1)\cdot}{\mathbf{q}}, t). \tag{2.1}$$

For each index $h = 0, 1, \dots, r + 1$ we consider the operator

$$\mathcal{O}_{r,h} = \sum_{j=0}^{h} (-1)^j \begin{pmatrix} r+1-h+j \\ r+1-h \end{pmatrix} \frac{d^j}{dt^j} \nabla_{\overset{(r+1-h+j)\cdot}{\mathbf{q}}} \tag{2.2}$$

which maps $\mathcal{Y}$ on $\mathbb{R}^{(r+2)\ell+1}$ on a ℓ–vector function depending on the $(r + 2 + h)\ell + 1$ independent variables $\mathbf{q}, \dot{\mathbf{q}}$, ..., $\overset{r\cdot}{\mathbf{q}}$, $\overset{(r+1)\cdot}{\mathbf{q}}$, ..., $\overset{(r+h+1)\cdot}{\mathbf{q}}$, t. For a fixed value of h the expression (2.2) contains $h + 1$ terms, which are worthy to be written explicitly, for greater clarity:

$$
\begin{array}{ll}
h = r+1: & \nabla_{\mathbf{q}}\mathcal{Y}_r - \frac{d}{dt}\nabla_{\dot{\mathbf{q}}}\mathcal{Y}_r + \frac{d^2}{dt^2}\nabla_{\ddot{\mathbf{q}}}\mathcal{Y}_r + \dots + (-1)^{r+1}\dfrac{d^{r+1}}{dt^{r+1}}\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\mathcal{Y}_r, \\
h = r: & \nabla_{\dot{\mathbf{q}}}\mathcal{Y}_r - 2\frac{d}{dt}\nabla_{\ddot{\mathbf{q}}}\mathcal{Y}_r + 3\frac{d^2}{dt^2}\nabla_{\dddot{\mathbf{q}}}\mathcal{Y}_r + \dots + (-1)^{r}(r+1)\frac{d^{r}}{dt^{r}}\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\mathcal{Y}_r, \\
& \dots \\
h: & \nabla_{\overset{(r+1-h)\cdot}{\mathbf{q}}}\mathcal{Y}_r - (r+1-h+1)\frac{d}{dt}\nabla_{\overset{(r+1-h+1)\cdot}{\mathbf{q}}}\mathcal{Y}_r + \dots \\
& \dots + (-1)^j \begin{pmatrix} r+1-h+j \\ r+1-h \end{pmatrix} \frac{d^j}{dt^j}\nabla_{\overset{(r+1-h+j)\cdot}{\mathbf{q}}}\mathcal{Y}_r + \dots \\
& \dots + (-1)^h \begin{pmatrix} r+1 \\ r+1-h \end{pmatrix} \frac{d^h}{dt^h}\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\mathcal{Y}_r, \\
& \dots \\
h = 1: & \nabla_{\overset{r\cdot}{\mathbf{q}}}\mathcal{Y}_r - (r+1)\frac{d}{dt}\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\mathcal{Y}_r, \\
h = 0: & \nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\mathcal{Y}_r.
\end{array}
\tag{2.3}
$$

The case $h = r+1$ needs a special comment: for $r = 0$ we get back to (1.1), for $r > 0$ the operator is the left hand side of the lagrangian equations of higher–order Lagrangians, formulated by Ostrogradski as early as in the 1850s; some current references for the study of this kind of higher order Lagrangian are Gay-Balmaz et al. [4], Gracia et al. [2], Nesterenko [5] and Simon [6]. At the same time, the generalized operator (2.2) is also implemented in the theoretical formulation of non–standard higher order constrained systems: some studies in this sense are, for instance, in Cendra and Grillo [7] and in Chen et al. [8].

Now, let $\bar{\mathbf{q}}(\mathbf{q}, t)$ be a $\mathcal{C}^\infty$–change of coordinates, with $det\, J_{\mathbf{q}}\bar{\mathbf{q}} \neq 0$: the relation (1.2) can be extended by successive derivations at any order $k \geq 1$:

$$
\overset{k\cdot}{\bar{\mathbf{q}}} = \overset{k\cdot}{\bar{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}, \dots, \overset{k\cdot}{\mathbf{q}}, t). \tag{2.4}
$$

Each of (2.4) exhibits a linear biunivocal correspondence between the variables at the highest order of derivation: indeed, (2.4) writes explicitly

$$
\overset{k\cdot}{\bar{\mathbf{q}}} = (J_{\bar{\mathbf{q}}}\mathbf{q})\overset{k\cdot}{\mathbf{q}} + \Phi_{k-1}(\mathbf{q}, \dot{\mathbf{q}}, \dots, \overset{(k-1)\cdot}{\mathbf{q}}, t)
$$

where

$$
\Phi_{k-1} = \sum_{j=0}^{k-2} \begin{pmatrix} k-1 \\ j \end{pmatrix} (J_{\mathbf{q}} \overset{(k-1-j)\cdot}{\bar{\mathbf{q}}}) \overset{(j+1)\cdot}{\mathbf{q}} + \frac{\partial}{\partial t} \overset{(k-1)\cdot}{\bar{\mathbf{q}}}
$$

(the formula is obtained by iterating the derivatives in (1.2) and by taking into account that $\dfrac{d^s}{dt^s} J_{\mathbf{q}}\bar{\mathbf{q}} = J_{\mathbf{q}} \overset{s\cdot}{\bar{\mathbf{q}}}$, $\dfrac{d^s}{dt^s}\dfrac{\partial \bar{\mathbf{q}}}{\partial t} = \dfrac{\partial}{\partial t}\overset{s\cdot}{\bar{\mathbf{q}}}$ for any $s \geq 0$).

Hence, for any given real–valued function $\mathcal{Y}(\mathbf{q}, \dot{\mathbf{q}}, \dots, \overset{r\cdot}{\mathbf{q}}, \overset{(r+1)\cdot}{\mathbf{q}}, t)$ we can compute $\widetilde{\mathcal{Y}}$ as the function $\mathcal{Y}$ calculated in the inverse transformations with respect to (2.4):

$$
\widetilde{\mathcal{Y}}(\bar{\mathbf{q}},, \dot{\bar{\mathbf{q}}}, \dots, \overset{r\cdot}{\bar{\mathbf{q}}}, \overset{(r+1)\cdot}{\bar{\mathbf{q}}}, t) = \mathcal{Y}(\mathbf{q}(\bar{\mathbf{q}}, t), \dot{\mathbf{q}}(\bar{\mathbf{q}}, \dot{\bar{\mathbf{q}}}, t), \dots, \overset{r\cdot}{\mathbf{q}}(\bar{\mathbf{q}}, \dot{\bar{\mathbf{q}}}, \dots, \overset{r\cdot}{\bar{\mathbf{q}}}, t), \overset{(r+1)\cdot}{\mathbf{q}}(\bar{\mathbf{q}}, \dot{\bar{\mathbf{q}}}, \dots, \overset{(r+1)\cdot}{\bar{\mathbf{q}}}, t), \tag{2.5}
$$

The considerable feature of (2.2) is that each of the $r+2$ operators abides by the same rule (1.3) of transformation: we state the following

Lemma 2.1. *For any $r \geq 0$ and any $h = 0, \dots, r+1$ it holds*

$$
\mathcal{O}_{r,h}[\mathcal{Y}] = (J_{\mathbf{q}}\bar{\mathbf{q}})^T \widetilde{\mathcal{O}}_{r,h}[\widetilde{\mathcal{Y}}] \tag{2.6}
$$

where $\widetilde{\mathcal{Y}}$ is computed as in (2.5) and $\widetilde{\mathcal{O}}_{r,h}$ is the operator (2.2) where each $\overset{(r+1-h+j)\cdot}{\mathbf{q}}$ is replaced by $\overset{(r+1-h+j)\cdot}{\bar{\mathbf{q}}}$, respectively, for $j = 0, \dots, h$.

Hint of the proof. One can proceed by induction on h: the case $h = 0$ is evident, since $\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\mathcal{Y} = (J_{\mathbf{q}}\bar{\mathbf{q}})^T \nabla_{\overset{(r+1)\cdot}{\bar{\mathbf{q}}}}\widetilde{\mathcal{Y}}$. Moreover, the case $h = 1$ will be checked in detail through next Section. The key point in order to conclude the proof is the formula

$$J_{\overset{u\cdot}{\mathbf{q}}}\overset{(u+s)\cdot}{\bar{\mathbf{q}}} = \begin{pmatrix} u+s \\ s \end{pmatrix} \frac{d^s}{dt^s} J_{\mathbf{q}}\bar{\mathbf{q}}, \quad s \geq 0 \tag{2.7}$$

(we already mentioned the case $u = 0$) which can be directly checked, or can be traced, for istance, in Craig [9]; the case $s = 0$ shows the known property of "cancelling dots" within the jacobian matrix of the transformation between variables showing the same number of dots. □

At this point, a relevant question could arise with regard to the role and the meaning that operators (1.3) can assume from the physical point of view, somehow extending to more general contexts the familiar situation $r = 0$, $h = 1$ for a Lagrangian $\mathcal{L}$ of a mechanical system.

As we already stated, the expression $h = r + 1$ is used in literature for higher–order Lagrangians, like the function $\mathcal{Y}$ we introduced. In what follows, we will focus on the case $h = 1$ which covers in our mind the generalization of the ordinary "Lagrangian binomial" $\mathcal{O}_{0,1}[\mathcal{L}] = \nabla_{\mathbf{q}}\mathcal{L} - \frac{d}{dt}(\nabla_{\dot{\mathbf{q}}}\mathcal{L})$. Our aim consists in analysing the properties of the operator and in associating such a case with concrete formulations of mechanical systems. An enlightening study of the case $h = 1$, $r = 1$ is performed in Minguzzi [10].

3 The Extended "Lagrangian Binomial"

For any fixed integer $r \geq 0$ let us consider the operator (2.2)

$$\mathcal{O}_r[\mathcal{Y}] = \nabla_{\overset{r\cdot}{\mathbf{q}}}\mathcal{Y} - (r+1)\frac{d}{dt}\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\mathcal{Y} \tag{3.1}$$

(where we dropped the second index $h = 1$ for simplicity) which maps a real–values function $\mathcal{Y}(\mathbf{q}, \dot{\mathbf{q}}, \dots, \overset{(r+1)\cdot}{\mathbf{q}}, t)$ on $\mathbb{R}^{(r+2)\ell+1}$ onto a ℓ–vector of functions in $\mathbb{R}^{(r+3)+1}$. As claimed before, we directly check the property (2.6) for this case:

Proposition 3.1. *For any fixed $r \geq 0$ and any function $\mathcal{Y}$ as in (6.4), the operator (3.1) verifies (2.6), that is $\mathcal{O}_r[\mathcal{Y}] = (J_{\bar{\mathbf{q}}}\mathbf{q})^T \widetilde{\mathcal{O}}_r[\widetilde{\mathcal{Y}}]$, where $\widetilde{\mathcal{Y}}$ is computed as in (2.5).*

Proof. We start from the calculation

$$\begin{aligned} \nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\mathcal{Y} &= \left(J_{\overset{(r+1)\cdot}{\mathbf{q}}}\overset{(r+1)\cdot}{\bar{\mathbf{q}}}\right)^T \nabla_{\overset{(r+1)\cdot}{\bar{\mathbf{q}}}}\widetilde{\mathcal{Y}}, \\ \nabla_{\overset{r\cdot}{\mathbf{q}}}\mathcal{Y} &= (J_{\overset{r\cdot}{\mathbf{q}}}\overset{r\cdot}{\bar{\mathbf{q}}})^T \nabla_{\overset{r\cdot}{\bar{\mathbf{q}}}}\widetilde{\mathcal{Y}} + \left(J_{\overset{r\cdot}{\mathbf{q}}}\overset{(r+1)\cdot}{\bar{\mathbf{q}}}\right)^T \nabla_{\overset{(r+1)\cdot}{\bar{\mathbf{q}}}}\widetilde{\mathcal{Y}} \end{aligned}$$

and we make use of the property (2.7) taking $s = 0$ and $s = 1$:

$$J_{\overset{(r+1)\cdot}{\mathbf{q}}}\overset{(r+1)\cdot}{\bar{\mathbf{q}}} = J_{\overset{r\cdot}{\mathbf{q}}}\overset{r\cdot}{\bar{\mathbf{q}}} = J_{\mathbf{q}}\bar{\mathbf{q}}, \quad J_{\overset{r\cdot}{\mathbf{q}}}\overset{(r+1)\cdot}{\bar{\mathbf{q}}} = (r+1)\tfrac{d}{dt}J_{\mathbf{q}}\bar{\mathbf{q}}.$$

Thus

$$
\begin{aligned}
&\nabla_{\overset{r\cdot}{\mathbf{q}}}\mathcal{Y} - (r+1)\tfrac{d}{dt}\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\mathcal{Y} \\
&= (J_{\mathbf{q}}\bar{\mathbf{q}})^T\nabla_{\overset{r\cdot}{\bar{\mathbf{q}}}}\widetilde{\mathcal{Y}} + (r+1)\tfrac{d}{dt}(J_{\mathbf{q}}\bar{\mathbf{q}})^T\nabla_{\overset{(r+1)\cdot}{\bar{\mathbf{q}}}}\widetilde{\mathcal{Y}} - (r+1)(J_{\mathbf{q}}\bar{\mathbf{q}})^T\tfrac{d}{dt}\nabla_{\overset{(r+1)\cdot}{\bar{\mathbf{q}}}}\widetilde{\mathcal{Y}} - (r+1)\tfrac{d}{dt}(J_{\mathbf{q}}\bar{\mathbf{q}})^T\nabla_{\overset{(r+1)\cdot}{\bar{\mathbf{q}}}}\widetilde{\mathcal{Y}} \\
&= (J_{\mathbf{q}}\bar{\mathbf{q}})^T\left(\nabla_{\overset{r\cdot}{\bar{\mathbf{q}}}}\widetilde{\mathcal{Y}}_r - (r+1)\tfrac{d}{dt}\nabla_{\overset{(r+1)\cdot}{\bar{\mathbf{q}}}}\widetilde{\mathcal{Y}}_r\right)
\end{aligned}
$$

whence the validity of (2.6) for the selected case, since $(J_{\mathbf{q}}\bar{\mathbf{q}})^{-1} = J_{\bar{\mathbf{q}}}\mathbf{q}$. □

The additional properties we need about the operator (3.1) concern the derivatives of a given function: the most evident is the following

Proposition 3.2. *For any $\mathcal{L}(\mathbf{q},\dot{\mathbf{q}},t)$ it holds $\mathcal{O}_0[\mathcal{L}] = \mathcal{O}_r\left[\frac{d^r\mathcal{L}}{dt^r}\right]$, that is*

$$
\nabla_{\mathbf{q}}\mathcal{L} - \frac{d}{dt}\nabla_{\dot{\mathbf{q}}}\mathcal{L} = \nabla_{\overset{r\cdot}{\mathbf{q}}}\frac{d^r\mathcal{L}}{dt^r} - (r+1)\frac{d}{dt}\left(\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\frac{d^r\mathcal{L}}{dt^r}\right). \tag{3.2}
$$

Proof. We first remark that $\dfrac{d^r\mathcal{L}}{dt^r} = \nabla_{\dot{\mathbf{q}}}\mathcal{L}\cdot\overset{(r+1)\cdot}{\mathbf{q}} + \mathbf{\Phi}(\mathbf{q},\dot{\mathbf{q}},\dots,\overset{r\cdot}{\mathbf{q}},t)$ hence $\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\dfrac{d^r\mathcal{L}}{dt^r} = \nabla_{\dot{\mathbf{q}}}\mathcal{L}$, so that (3.2) is equivalent to

$$
\nabla_{\overset{r\cdot}{\mathbf{q}}}\frac{d^r\mathcal{L}}{dt^r} = \nabla_{\mathbf{q}}\mathcal{L} + r\frac{d}{dt}\nabla_{\dot{\mathbf{q}}}\mathcal{L}. \tag{3.3}
$$

For $r=1$ it is immediate to check that $\nabla_{\dot{\mathbf{q}}}\frac{d\mathcal{L}}{dt} = \nabla_{\mathbf{q}}\mathcal{L} + \frac{d}{dt}\nabla_{\dot{\mathbf{q}}}\mathcal{L}$; for $r>1$ we proceed by induction on r, assuming that (3.3) holds. Let us take advantage of the property concerning the inversion between total derivative and gradient:

$$
\nabla_{\overset{n\cdot}{\mathbf{q}}}\frac{dF}{dt} - \frac{d}{dt}\nabla_{\overset{n\cdot}{\mathbf{q}}}F = \nabla_{\overset{(n-1)\cdot}{\mathbf{q}}}F \tag{3.4}
$$

which is valid for any real values–function $F(\mathbf{q},\dot{\mathbf{q}},\dots,\overset{n\cdot}{\mathbf{q}},t)$ and any $n\geq 1$. By employing (3.4) with $F=\frac{d^r\mathcal{L}}{dt^r}$ and $n=r+1$ one achieves

$$
\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\frac{d^{r+1}\mathcal{L}}{dt^{r+1}} = \nabla_{\overset{r\cdot}{\mathbf{q}}}\frac{d^r\mathcal{L}}{dt^r} + \frac{d}{dt}\left(\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\frac{d^r\mathcal{L}}{dt^r}\right) = \nabla_{\overset{r\cdot}{\mathbf{q}}}\frac{d^r\mathcal{L}}{dt^r} + \frac{d}{dt}\nabla_{\dot{\mathbf{q}}}\mathcal{L}
$$

and, by virtue of the induction assumption (3.3),

$$
\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\frac{d^{r+1}\mathcal{L}}{dt^{r+1}} = \nabla_{\mathbf{q}}\mathcal{L} + (r+1)\frac{d}{dt}\nabla_{\dot{\mathbf{q}}}\mathcal{L}
$$

hence (3.3) holds for $r+1$, too. □

We render now relation (3.4) more general by means of the following

Proposition 3.3. *For any function $\mathcal{U}(\mathbf{q},\dot{\mathbf{q}},\dots,\overset{s\cdot}{\mathbf{q}},\overset{(s+1)\cdot}{\mathbf{q}},t)$, $s\geq 0$ and any index $r\geq s$ it is*

$$
\mathcal{O}_r\left[\frac{d^{r-s}\mathcal{U}}{dt^{r-s}}\right] = \mathcal{O}_s[\mathcal{U}]. \tag{3.5}
$$

Proof. It has to be checked that

$$
\nabla_{\overset{r\cdot}{\mathbf{q}}}\frac{d^{r-s}\mathcal{U}}{dt^{r-s}} - (r+1)\frac{d}{dt}\left(\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\frac{d^{r-s}\mathcal{U}}{dt^{r-s}}\right) = \nabla_{\overset{s\cdot}{\mathbf{q}}}\mathcal{U} - (s+1)\frac{d}{dt}\nabla_{\overset{(s+1)\cdot}{\mathbf{q}}}\mathcal{U}. \tag{3.6}
$$

Since $\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}} \frac{d^{r-s}\mathcal{U}}{dt^{r-s}} = \nabla_{\overset{(s+1)\cdot}{\mathbf{q}}}\mathcal{U}$, (3.6) can be written as

$$\nabla_{\overset{r\cdot}{\mathbf{q}}} \frac{d^{r-s}\mathcal{U}}{dt^{r-s}} = (r-s)\frac{d}{dt}\nabla_{\overset{(s+1)\cdot}{\mathbf{q}}}\mathcal{U} + \nabla_{\overset{s\cdot}{\mathbf{q}}}\mathcal{U}. \tag{3.7}$$

We refer again to (3.4) which is used $r-s$ times for $F = \dfrac{d^{r-s-1}\mathcal{U}}{dt^{r-s-1}}$, $F = \dfrac{d^{r-s-2}\mathcal{U}}{dt^{r-s-2}}$, $\dots$, $F = \mathcal{U}$ and $n = r$, $r-1$, $\dots$, $r-(r-s)+1 = s+1$ respectively:

$$\nabla_{\overset{r\cdot}{\mathbf{q}}} \frac{d^{r-s}\mathcal{U}}{dt^{r-s}} = \nabla_{\overset{r\cdot}{\mathbf{q}}}\left(\frac{d}{dt}\frac{d^{r-s-1}\mathcal{U}}{dt^{r-s-1}}\right) = \frac{d}{dt}\left(\nabla_{\overset{r\cdot}{\mathbf{q}}}\frac{d^{r-s-1}\mathcal{U}}{dt^{r-s-1}}\right) + \nabla_{\overset{(r-1)\cdot}{\mathbf{q}}}\frac{d^{r-s-1}\mathcal{U}}{dt^{r-s-1}},$$

$$\nabla_{\overset{(r-1)\cdot}{\mathbf{q}}} \frac{d^{r-s-1}\mathcal{U}}{dt^{r-s-1}} = \nabla_{\overset{(r-1)\cdot}{\mathbf{q}}}\left(\frac{d}{dt}\frac{d^{r-s-2}\mathcal{U}}{dt^{r-s-2}}\right) = \frac{d}{dt}\left(\nabla_{\overset{(r-1)\cdot}{\mathbf{q}}}\frac{d^{r-s-2}\mathcal{U}}{dt^{r-s-2}}\right) + \nabla_{\overset{(r-2)\cdot}{\mathbf{q}}}\frac{d^{r-s-2}\mathcal{U}}{dt^{r-s-2}},$$

$$\dots \quad \dots \quad \dots$$

$$\nabla_{\overset{(s+1)\cdot}{\mathbf{q}}}\frac{d}{dt}\mathcal{U} = \frac{d}{dt}\nabla_{\overset{(s+1)\cdot}{\mathbf{q}}}\mathcal{U} + \nabla_{\overset{s\cdot}{\mathbf{q}}}\mathcal{U}.$$

Since

$$\nabla_{\overset{(r-1)\cdot}{\mathbf{q}}}\left(\frac{d^{r-s-1}\mathcal{U}}{dt^{r-s-1}}\right) = \nabla_{\overset{(r-1)\cdot}{\mathbf{q}}}\left(\frac{d^{r-s-2}\mathcal{U}}{dt^{r-s-2}}\right) = \dots = \nabla_{\overset{(s+1)\cdot}{\mathbf{q}}}\mathcal{U},$$

relation (3.7) easily follows. We remark that (3.2) is (3.6) whenever $s = 0$. □

4 Extending Kinetic Energy and Potential Forces

In order to give a real physical context to the operator (3.1) a nd to place in a right way the properties discussed above, we introduce a system of N material points $\{P_i, m_i\}_{i=1,\dots,N}$. If (x_i, y_i, z_i) are the cartesian coordinates of the i–point with respect to a fixed frame of reference, we list all of them orederly in a $3n$–vector $\mathbf{X} \in \mathbb{R}^{3N}$. Calling $\mathbf{L} = (m_1\dot{P}_1, \dots, m_N\dot{P}_N)$ the $3N$–vector of the linear momentum of the system, we define

$$\mathcal{T}_r = \frac{1}{2}\overset{r\cdot}{\mathbf{L}} \cdot \overset{(r+1)\cdot}{\mathbf{X}}. \tag{4.1}$$

For $r = 0$ (4.1) is the standard kinetic energy $\mathcal{T}_0 = \frac{1}{2}\mathbf{L}\cdot\dot{\mathbf{X}} = \frac{1}{2}\sum\limits_{i=1}^{N} m_i\dot{P}_i^2$. A second case to be remarked is $r = 1$, providing the *acceleration energy* (see Neimark and Fufaev [11]):

$$\mathcal{T}_1 = \frac{1}{2}\dot{\mathbf{L}}\cdot\ddot{\mathbf{X}} = \frac{1}{2}\sum_{i=1}^{N} m_i\ddot{P}_i^2. \tag{4.2}$$

Now, if the system of points is subjected to $m < 3N$ independent holonomic constraints, the coordinates $\mathbf{X}$ turn out to be expressed by $\ell = 3N - m$ lagrangian coordinates according to $\mathbf{X}(\mathbf{q}, t)$, $\mathbf{q} \in \mathbb{R}^\ell$ (the dependence on t is due to possible rheonomic constraints). The function $\mathcal{T}_r$ of (4.1) is now $\mathcal{T}_r = \mathcal{T}_r(\mathbf{q}, \dot{\mathbf{q}}, \dots, \overset{r\cdot}{\mathbf{q}}, \overset{(r+1)\cdot}{\mathbf{q}}, t)$. The well known relation between the lagrangian components of $\mathbf{L}$ and (3.1) regarding the case $r = 0$ can be expanded to higher derivatives by means of the following

Proposition 4.1.

$$\overset{(r+1)\cdot}{\mathbf{L}} \cdot \frac{\partial \mathbf{X}}{\partial q_k} = \frac{d}{dt}\frac{\partial \mathcal{T}_r}{\partial \overset{(r+1)\cdot}{q_k}} - \frac{1}{r+1}\frac{\partial \mathcal{T}_r}{\partial \overset{r\cdot}{q_k}}, \quad k = 1, \dots, \ell \tag{4.3}$$

Proof. It suffices to replicate the steps of the standard case $r=0$, by writing

$$\overset{(r+1)\cdot}{\mathbf{L}} \cdot \frac{\partial \mathbf{X}}{\partial q_k} = \frac{d}{dt}\overset{r\cdot}{\mathbf{L}} \cdot \frac{\partial \mathbf{X}}{\partial q_k} - \overset{r\cdot}{\mathbf{L}} \cdot \frac{\partial \dot{\mathbf{X}}}{\partial q_k}, \quad k=1,\dots,\ell.$$

On the other hand, one has

$$\frac{\partial \mathcal{T}_r}{\partial \overset{(r+1)\cdot}{q_k}} = \overset{r\cdot}{\mathbf{L}} \cdot \frac{\partial \overset{(r+1)\cdot}{\mathbf{X}}}{\partial \overset{(r+1)\cdot}{q_k}} = \overset{r\cdot}{\mathbf{L}} \cdot \frac{\partial \mathbf{X}}{\partial q_k}, \qquad \frac{\partial \mathcal{T}_r}{\partial \overset{r\cdot}{q_k}} = \overset{r\cdot}{\mathbf{L}} \cdot \frac{\partial \overset{(r+1)\cdot}{\mathbf{X}}}{\partial \overset{r\cdot}{q_k}} = (r+1)\overset{r\cdot}{\mathbf{L}} \cdot \frac{\partial \mathbf{X}}{\partial q_k}$$

where the second equalities in each of the two sequences are deduced from relations analogous to (2.7). □

Remark 4.1. In the case $r=1$, the acceleration energy (4.2) verifies

$$\frac{\partial \mathcal{T}_1}{\partial \ddot{q}_k} = \dot{\mathbf{L}} \cdot \frac{\partial \mathbf{X}}{\partial q_k}$$

and the Newton's law makes us write the equations of motion in the form

$$\frac{\partial \mathcal{T}_1}{\partial \ddot{q}_k} = \mathcal{F}_k, \quad k=1,\dots,\ell$$

where $\mathcal{F}_k$ is the k–lagrangian component of the force. The just written equations are the Appell equations (for which Whittaker [12] is a historical reference), in the special case of pseudovelocities matching with the generalized velocities.

Assume now that the system of forces $\boldsymbol{\mathcal{F}} \in \mathbb{R}^{3N}$ exerting on $(P_1,\dots,P_N)$ admits a function $\mathcal{U}(\mathbf{q}, \dot{\mathbf{q}}, \dots, \overset{(s+1)\cdot}{\mathbf{q}}, t)$, $s \geq 0$, such that

$$\boldsymbol{\mathcal{F}} \cdot \frac{\partial \mathbf{X}}{\partial q_k} = \frac{\partial \mathcal{U}}{\partial \overset{s\cdot}{q_k}} - (s+1)\frac{d}{dt}\frac{\partial \mathcal{U}}{\partial \overset{(s+1)\cdot}{q_k}}. \tag{4.4}$$

The case $s=0$ corresponds to a force connected to a generalized potential $\mathcal{U}(\mathbf{q}, \dot{\mathbf{q}}, t)$hl: some relevant instances are the Lorentz force in an electromagnetic field or the couple Coriolis force–centrifugal force exerted by a reference frame in uniform rotation. Both circumstances of Lorentz force and non inertial forces fall in the case of linear dependence on the velocity

$$\boldsymbol{\mathcal{F}}(\mathbf{q}, \dot{\mathbf{q}}, t) = \left(J_{\mathbf{q}}^T \boldsymbol{\alpha} - J_{\mathbf{q}} \boldsymbol{\alpha}\right)^T \dot{\mathbf{q}} + \nabla_{\mathbf{q}} \beta - \frac{\partial \boldsymbol{\alpha}}{\partial t}$$

(with appropriate $\boldsymbol{\alpha}$ and β) combined with the potential $\mathcal{U}_1(\mathbf{q}, \dot{\mathbf{q}}, t) = \boldsymbol{\alpha}(\mathbf{q}, t) \cdot \dot{\mathbf{q}} + \beta(\mathbf{q}, t)$.

In order to improve notations, if one introduces the matrix–vector product $(J_{\mathbf{q}}\mathbf{X})^T \mathbf{w} = \begin{pmatrix} \mathbf{w} \cdot \frac{\partial \mathbf{X}}{\partial q_1} \\ \dots \\ \mathbf{w} \cdot \frac{\partial \mathbf{X}}{\partial q_\ell} \end{pmatrix}$ providing the ℓ lagrangian components of any $3n$–vector $\mathbf{w}$, (4.3) and (4.4) can be written in terms of the operator (3.1) respectively as

$$(J_{\mathbf{q}}\mathbf{X})^T \overset{(r+1)\cdot}{\mathbf{L}} = -\frac{1}{r+1}\mathcal{O}_r[\mathcal{T}_r], \quad (J_{\mathbf{q}}\mathbf{X})^T \mathcal{F} = \mathcal{O}_s[\mathcal{U}]. \tag{4.5}$$

It is also worthy to remark that (4.3) allows us to place in the context of (3.1) special forces proportional to the acceleration, or the rate of change of acceleration or even further derivatives with respect to time:

$$\boldsymbol{\mathcal{F}}_A = \mathbb{L} \overset{(h+1)\cdot}{\mathbf{X}} \tag{4.6}$$

where $h \geq 1$ and $\mathbb{L}$ is a diagonal matrix with positive entries. Calculations very close to those concerning (4.3) lead to

$$\mathbb{L}\overset{(h+1)\cdot}{\mathbf{X}}\cdot\frac{\partial\mathbf{X}}{\partial q_k}=\frac{d}{dt}\frac{\partial\mathcal{U}_A}{\partial\overset{h\cdot}{q_k}}-\frac{1}{h}\frac{\partial\mathcal{U}_A}{\partial\overset{(h-1)\cdot}{q_k}},\quad k=1,\dots,\ell$$

where $\mathcal{U}_A=\frac{1}{2}\mathbb{L}\overset{h\cdot}{\mathbf{X}}\cdot\overset{h\cdot}{\mathbf{X}}$. According to the notation of (3.1) we can also write

$$(J_{\mathbf{q}}\mathbf{X})^T\boldsymbol{\mathcal{F}}_A=-\frac{1}{h}\mathcal{O}_{h-1}[\mathcal{U}_A]. \tag{4.7}$$

For $h=1$ and $\mathbb{L}$ listing the masses of the points one recovers the inertial forces. A second circumstance to have in mind is the Abraham–Lorentz force (due to the electromagnetic radiation on an accelerating charged particle), which can be modeled choosing $h=2$:

$$\boldsymbol{\mathcal{F}}_A=\mathbb{L}\,\ddot{\mathbf{X}},\quad \mathcal{U}_A=\frac{1}{2}\mathbb{L}\,\ddot{\mathbf{X}}\cdot\ddot{\mathbf{X}}$$

where the diagonal matrix $\mathbb{L}$ depends on the charge of the particle, the speed of light and on the electric and magnetic constants. The lagrangian components of the force verify in this case

$$\boldsymbol{\mathcal{F}}_A\cdot\frac{\partial\mathbf{X}}{\partial q_k}=\frac{d}{dt}\frac{\partial\mathcal{U}_A}{\partial\ddot{q}_k}-\frac{1}{2}\frac{\partial\mathcal{U}_A}{\partial\dot{q}_k}.$$

5 Other Types of Forces

Let us make a very simple point: for any real values–function $\gamma(\mathbf{q},\dot{\mathbf{q}},\dots,\overset{i\cdot}{\mathbf{q}},t)$ and any index $h\geq 1$ it holds

$$\mathcal{O}_i[\gamma]=\nabla_{\overset{i\cdot}{\mathbf{q}}}\gamma. \tag{5.1}$$

This allows us to count the following cases in the formal stucture (3.1):

(i) a system of forces $\boldsymbol{\mathcal{F}}_R$ such that the lagrangian components verify

$$(J_{\mathbf{q}}\mathbf{X})^T\boldsymbol{\mathcal{F}}_R=-\nabla_{\overset{\sigma\cdot}{\mathbf{q}}}\mathcal{R}(\mathbf{q},\dot{\mathbf{q}},\dots,\overset{\sigma\cdot}{\mathbf{q}},t) \tag{5.2}$$

for some function $\mathcal{R}$ and some positive integer σ;

(ii) a general force $\boldsymbol{\mathcal{G}}(\mathbf{q},\dot{\mathbf{q}},\dots,\overset{(\varrho-1)\cdot}{\mathbf{q}},t)$, $\varrho\geq 1$.

In the first case, by virtue of (5.1) we have

$$\mathcal{O}_\sigma[-\mathcal{R}]=(J_{\mathbf{q}}\mathbf{X})^T\boldsymbol{\mathcal{F}}. \tag{5.3}$$

The signus $-$ is appropriate if we think of the case $\sigma=1$, producing the dissipation force $\boldsymbol{\mathcal{F}}_R=-(J_{\mathbf{q}}\mathbf{X}_\mu)\dot{\mathbf{q}}$, where $\mathbf{X}_\mu=(\mu_1P_1,\dots,\mu_nP_n)$ (μ_i friction coefficients) is a practical notation. The lagrangian components are $(J_{\mathbf{q}}\mathbf{X})^T\boldsymbol{\mathcal{F}}_R=-\mathbb{D}\dot{\mathbf{q}}$, being $\mathbb{D}(\mathbf{q},t)=(J_{\mathbf{q}}\mathbf{X})^T J_{\mathbf{q}}\mathbf{X}_\mu$ symmetric and positive definite, and $\mathcal{R}=-\frac{1}{2}\dot{\mathbf{q}}\cdot\mathbb{D}\dot{\mathbf{q}}$. Condition (5.2) writes in this case $\nabla_{\dot{\mathbf{q}}}\mathcal{R}=-\mathbb{D}\dot{\mathbf{q}}$.

With regard to case (ii), the device of defining the function $\gamma_\varrho(\mathbf{q},\dot{\mathbf{q}},\dots,\overset{\varrho\cdot}{\mathbf{q}},t)=(J_{\mathbf{q}}\mathbf{X})^T\boldsymbol{\mathcal{G}}\cdot\overset{\varrho\cdot}{\mathbf{q}}$ provides, according to (5.1),

$$\mathcal{O}_\varrho[\gamma_\varrho]=(J_{\mathbf{q}}\mathbf{X})^T\boldsymbol{\mathcal{G}}. \tag{5.4}$$

In this way the lagrangian components of a generic force $\boldsymbol{\mathcal{G}}(\mathbf{q},\dots,\overset{(\varrho-1)\cdot}{\mathbf{q}},t)$ can be intercepted by the operator (3.1) by means of a function γ_ϱ, at the expense of expanding the variables to the subsequent derivative $\overset{\varrho\cdot}{\mathbf{q}}$.

6 Equations of Motion

We are going now to assemble the various aspects: the main objective is to encompass contributions of different kinds under the one equation (3.1), for appropriate r and $\mathcal{Y}_r$.

Basing on the Newton's law splitted on the lagrangian components

$$(J_{\mathbf{q}}\mathbf{X})^T\dot{\mathbf{L}} = (J_{\mathbf{q}}\mathbf{X})^T\boldsymbol{\mathcal{F}}, \tag{6.1}$$

we model the dynamical term $\boldsymbol{\mathcal{F}}$. We separate the contributions according to

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_0 + \boldsymbol{\mathcal{F}}_S + \boldsymbol{\mathcal{F}}_A + \boldsymbol{\mathcal{F}}_R + \boldsymbol{\mathcal{G}} \tag{6.2}$$

exhibiting the following features:

(*i*) $\boldsymbol{\mathcal{F}}_0$ originates from a potential $U_0(\mathbf{q}, t)$ such that $(J_{\mathbf{q}}\mathbf{X})^T\boldsymbol{\mathcal{F}}_0 = \nabla_{\mathbf{q}} U_0$,

(*ii*) the term $\boldsymbol{\mathcal{F}}_S$ corresponds to a generalized potential and verifies (4.4) for some $s \geq 0$, so that $(J_{\mathbf{q}}\mathbf{X})^T\boldsymbol{\mathcal{F}}_S = \mathcal{O}_s[\mathcal{U}_s]$ for an appropriate $\mathcal{U}_s$,

(*iii*) the contribution $\boldsymbol{\mathcal{F}}_A$ is caused by acceleration effects or by higher order effects and it accounts for condition (4.6) with a suitable integer $h \geq 1$; the function (4.7) provides the lagrangian components of the force according to $\mathcal{O}_{h-1}[\mathcal{U}_A] = -h(J_{\mathbf{q}}\mathbf{X})^T\boldsymbol{\mathcal{F}}_A$;

(*iv*) the term $\boldsymbol{\mathcal{F}}_R$ is of the (5.2) type for some $\sigma \geq 1$ and the lagrangian components verify (5.3),

(*v*) the contribution $\boldsymbol{\mathcal{G}}$ concerns a generic force depending on $\mathbf{q}$, $\dot{\mathbf{q}}$, ..., $\overset{(\varrho-1)\cdot}{\mathbf{q}}$ and possibly t for some $\varrho \geq 1$: the formal settlement to have in mind is (5.4).

For a given system of forces (6.2) we set $r = \max(s, m-1, \sigma, \varrho)$. On the ground of the previous analysis, we are going to prove the following

Proposition 6.1. *The equations of motion for the set of material points* (m_i, P_i), $i = 1, \ldots, N$ *subject to the system of forces (6.2) are*

$$\mathcal{O}_r[\mathcal{Y}_r] = \nabla_{\overset{r\cdot}{\mathbf{q}}}\mathcal{Y}_r - (r+1)\frac{d}{dt}\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\mathcal{Y}_r = 0 \tag{6.3}$$

where

$$\mathcal{Y}_r = \frac{d^r\mathcal{L}}{dt^r} + \frac{d^{r-s}U_S}{dt^{r-s}} - \frac{1}{h}\frac{d^{r-(h-1)}U_A}{dt^{r-(h-1)}} - \frac{d^{r-\sigma}\mathcal{R}}{dt^{r-\sigma}} + \frac{d^{r-\varrho}\gamma_\varrho}{dt^{r-\varrho}} \tag{6.4}$$

and $\mathcal{L} = \frac{1}{2}\mathbf{L}\cdot\dot{\mathbf{X}} + U_0$, *where* $\mathbf{L}$ *is the* $3N$*–vector of the linear momenta* $m_i\dot{P}_i$, $i = 1, \ldots, N$.

Proof: By virtue of Propositions 1.2, 1.3 and the linearity of (3.1) we can write

$$\mathcal{O}_r[\mathcal{Y}_r] = \mathcal{O}_0[\mathcal{L}] + \mathcal{O}_s[\mathcal{U}_S] - \frac{1}{m}\mathcal{O}_{m-1}[\mathcal{U}_A] - \mathcal{O}_\sigma[\mathcal{R}] + \mathcal{O}_\varrho[\gamma_\varrho].$$

On the one hand, the first of (4.5) calculated for $r = 0$ means $\mathcal{O}_0[\mathcal{L}] = -(J_{\mathbf{q}}\mathbf{X})^T\dot{\mathbf{L}} + \nabla_{\mathbf{q}}U_0$; on the other hand, the second in (4.5) and (4.7), (5.3), (5.4) entail

$$\mathcal{O}_s[\mathcal{U}_S] - \frac{1}{m}\mathcal{O}_{m-1}[\mathcal{U}_A] - \mathcal{O}_\sigma[\mathcal{R}] + \mathcal{O}_\varrho[\gamma_\varrho] = (J_{\mathbf{q}}\mathbf{X})^T(\boldsymbol{\mathcal{F}}_S + \boldsymbol{\mathcal{F}}_A + \boldsymbol{\mathcal{F}}_R + \boldsymbol{\mathcal{G}})$$

so that (6.3) is equivalent to $(J_{\mathbf{q}}\mathbf{X})^T(-\dot{\mathbf{Q}} + \boldsymbol{\mathcal{F}}) = 0$, that is (6.1). □

A significant example for the extended formalism is the already mentioned Abraham–Lorentz force (see Rohrlich [13] for a review), where the term $\frac{2}{3}e\,\ddot{\mathbf{v}}$ is added to the electomagnetic Lorentz force $e(\mathbf{E}+\mathbf{v}\wedge\mathbf{B})$: the latter term is of type (*ii*), with $s = 0$, while the first term is of type (*iii*), $h = 2$ (see also (4.6)).The arrangement of the Abraham–Lorentz force under Lagrangian equations of motion is in Carati and Galgani [14].

In the end, we remark that combining (3.1) with (3.4) calculated for $n = r + 1$ and $P = \mathcal{Y}_r$, one finds

$$\mathcal{O}_r[\mathcal{Y}_r] = \nabla_{\overset{(r+1)\cdot}{\mathbf{q}}} \frac{d\mathcal{Y}_r}{dt} - (r+2)\frac{d}{dt}\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\mathcal{Y}_r. \tag{6.5}$$

Whenever $\mathcal{O}_r[\mathcal{Y}_r] = 0$ gives the equations of motion, by replacing $\frac{d}{dt}\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}}\mathcal{Y}_r$ with $\frac{1}{r+1}\nabla_{\overset{r\cdot}{\mathbf{q}}}\mathcal{Y}_r$ in (6.5) one achieves

$$\nabla_{\overset{(r+1)\cdot}{\mathbf{q}}} \frac{d\mathcal{Y}_r}{dt} - \frac{r+2}{r+1}\nabla_{\overset{r\cdot}{\mathbf{q}}}\mathcal{Y}_r = 0 \tag{6.6}$$

which can be considered a sort of *generalized Nielsen's equations*, in the sense that the case $r = 0$

$$\nabla_{\dot{\mathbf{q}}} \frac{d\mathcal{L}}{dt} - 2\nabla_{\mathbf{q}}\mathcal{L} = 0 \tag{6.7}$$

for $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ corresponds to the set known as the *Nielsen form of the equations of motion* (a relatively recent study is in Wang and Mei [15]).

7 Conclusions

The covariance property (2.6) of the binomial expression (3.1) with respect to point transformations provides access to make use of it for the motion of a systems under wide assumtions about the dynamical strains. The physical motivation of the higher order operator passes through the Newton's law and the lagrangian components of an overall force (6.2). The latter one includes generalized potentials (4.4), kinematic effects (4.6), generalized dissipative forces (5.2) or even non structured forces. For specific values of the indexes, the standard equations for the known and ordinary cases are reproduced.

By means of the property (3.5), all these effects can be encompassed by the function $\mathcal{Y}_r$, where r depends on the highest order of derivative occurring in each contribution. The order of the equations of motion (6.3) is $r + 2$. As (6.4) shows, the unifying procedure possibly requires derivations on the individual terms: on the other hand, the compact expression (6.3) may facilitate the investigation on specific properties of the system, in the same way as the energy balance can be deduced from the ordinary lagrangian equations of motion.

At the same time it is interesting to detect that some alternative form of the equations of motion, involving the derivatives of state functions (Nielsen or Appell equations) can be recovered from the set of equations (6.3).

An interesting further development will be the investigation of operators (2.2), for $h \neq 1$, especially with regard to their bearing on real physical systems. On the other hand, the context is well suited for letting more the constraint equations more general, embracing linear and nonlinear holonomic restrictions.

Competing Interests

Author has declared that no competing interests exist.

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