Asian Journal of Advanced Research and Reports

14(4): 131-141, 2020; Article no.AJARR.63223
ISSN: 2582-3248

A Note Concerning Generalized Complex *k*-Horadam and Generalized Gaussian *k*-Horadam Squences

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Authors' contributions

This work was carried out in collaboration between both authors. Author HG designed the study, performed the statistical analysis, wrote the protocol, wrote the first draft of the manuscript and managed the analyses of the study. Author HK managed the literature searches. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/AJARR/2020/v14i430350
Editor(s):
(1) Dr. Fagbadebo Omololu Michael, Durban University of Technology, South Africa.
Reviewers:
(1) Roniere Leite Soares, Federal University of Campina Grande, Brazil.
(2) Tomasz Rak, Rzeszow University of Technology, Poland.
(3) Jih-Fu Tu, Lunghwa University of Science and Technology, Taiwan.
Complete Peer review History: http://www.sdiarticle4.com/review-history/63223

Original Research Article

Received 04 October 2020
Accepted 11 December 2020
Published 29 December 2020

ABSTRACT

In this paper, we have studied the generalized Complex *k*-Horadam and generalized Gaussian *k*-Horadam sequences. We have given the generating functions and Binet formulas of generalized Complex *k*-Horadam and generalized Gaussian *k*-Horadam sequences. Moreover, we have obtained some important identities involving the generalized Complex *k*-Horadam and generalized Gaussian *k*-Horadam numbers.

Keywords: Generalized Complex k-Horadam; generalized Gaussian k-Horadam; Binet formula; d'Ocagne formula; Catalan formula.

2010 Mathematics Subject Classification: 11B37, 11B99, 11B83.

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1. INTRODUCTION

Number sequences have many applications and play an important role in the field of biology, physics, chemistry, computer science, architecture and art. As in computation of spline functions, time series analysis, signal and image processing, queueing theory, polynomial and power series computations and many other areas, typical problems modelled by Toeplitz matrices are the numerical solution of certain differential and integral equations [1,2,3]. The literature includes many papers dealing with the special number sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Mersenne, Fermat, Padovan and Perrin [4-20]. The books written by Koshy and Vajda collect and classify many results dealing with these number sequences, most of which have been obtained quite recently [16,18,19]. These number sequences have been generalized in many ways. Some equalities and inequalities have been found to be important for these numbers.

Yazlık and Taşkara [20] defined a generalization *k*-Horadam sequence of the special second order sequences such as Fibonacci, Lucas, Pell, Jacobsthal and Horadam. Also, they investigated some of the new algebraic proporties for the generalized *k*-Horadam sequence. Gökbas¸ and Köse [10] defined generalized comlex *k*-Horadam and generalized Gaussian k-Horadam sequences corresponding to the number sequences in the literature such as complex Fibonacci, complex Lucas, Gaussian Pell and Gaussian Jacobsthal. In addition, they gave some formulas for these sequences.

In this paper, we have studied generalized complex *k*-Horadam and generalized Gaussian *k*-Horadam sequences. We have given generating functions and Binet formulas for these sequences. Moreover, we have obtained well-known Cassini, Catalan and d'Ocagne identities involving generalized complex *k*-Horadam and generalized Gaussian *k*-Horadam numbers.

In the section below, we have handle some definitions to use as a result of this article.

Definition 1. [20] Let k be any positive real number and $f(k)$, $g(k)$ are scaler-value polynomials. For $n \geq 0$ and $f^2(k)+4g(k)>0$, the generalized *k*-Horadam sequence $\{H_{k,n}\}_{n\in N}$ is defined by

$$H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n}$$

with initial conditions $H_{k,0} = a$ and $H_{k,1} = b$.

In Table 1 for definition 1 summarizes some special cases of $f(k)$, $g(k)$, $H_{k,0}$ and $H_{k,1}$.

By giving the $f(k)$, $g(k)$, $H_{k,0}$ and $H_{k,1}$ values, the number sequences in the literature can be reached.

Definition 2. [20] Let r_1 and r_2 be the roots of a second order difference equation of generalized *k*-Horadam sequence with recurrence relation $H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n}$. For every $n \in N$

$$H_{k,n} = r_1 H_{k,n-1} + (H_{k,1} - r_1 H_{k,0}) r_2^{n-1}.$$

Definition 3. [20] Let r_1 and r_2 be the roots of a second order difference equation of generalized *k*-Horadam sequence with recurrence relation $H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n}$. For every $n \in N$, the Binet formula

$$H_{k,n} = \frac{(H_{k,1} - r_2 H_{k,0}) r_1^n - (H_{k,1} - r_1 H_{k,0}) r_2^n}{r_1 - r_2}.$$

Table 1. Some special cases of $f(k)$, $g(k)$, $H_{k,0}$ and $H_{k,1}$

$[f(k), g(k)]$	$(H_{k,0}, H_{k,1})$	***k*-Horadam Numbers**
[1, 1]	(0, 1)	Fibonacci Numbers
[2, 1]	(2, 2)	Pell-Lucas Numbers
[1, 2]	(2, 1)	Jacobsthal-Lucas Numbers
[*k*, 1]	(0, 1)	*k*-Fibonacci Numbers

Definition 4. [20] Let $n, m, r \geq 0$ and $\{H_{k,n}\}$ the generalized *k*-Horadam sequence be.

$$H_{k,m}H_{k,n} - H_{k,m+r}H_{k,n-r} = \frac{[-g(k)]^{n-r}(H_{k,1}H_{k,r} - H_{k,0}H_{k,r+1})(H_{k,1}H_{k,m-n+r} - H_{k,0}H_{k,m-n+r+1})}{H_{k,1}^2 - H_{k,0}^2 g(k) - H_{k,0}H_{k,1}f(k)}.$$

In the section below, we have handle some definitions to use as a result of this article (to be thrown, written twice)

Definition 5. [10] Let k be any positive real number and $f(k)$, $g(k)$ are scaler-value polynomials. For $n \geq 0$ and the generalized *k*-Horadam sequence, the generalized Complex *k*-Horadam sequence $\{CH_{k,n}\}_{n \in N}$ is defined by

$$CH_{k,n+2} = [f(k) + if^2(k) + ig(k)]H_{k,n+1} + [g(k) + if(k)g(k)]H_{k,n}$$

with initial conditions $H_{k,0} = a$ and $H_{k,1} = b$ and where i is the imaginary unit which satisfies $i^2 = -1$. When equality is arranged,

$$CH_{k,n+2} = H_{k,n+2} + iH_{k,n+3}.$$

In Table 2 for definition 5 summarizes some special cases of $f(k)$, $g(k)$, $H_{k,0}$ and $H_{k,1}$.

By giving the $f(k)$, $g(k)$, $H_{k,0}$ and $H_{k,1}$ values, the number sequences in the literature can be reached.

Definition 6. [10] Let k be any positive real number and $f(k)$, $g(k)$ are scaler-value polynomials. For $n \geq 0$ and the generalized *k*-Horadam sequence, the generalized Gaussian *k*-Horadam sequence $\{GH_{k,n}\}_{n \in N}$ is defined by

$$GH_{k,n+2} = [f^2(k) + g(k) + if(k)]H_{k,n} + [f(k)g(k) + ig(k)]H_{k,n-1}$$

with initial conditions $H_{k,0} = a$ and $H_{k,1} = b$ and where i is the imaginary unit which satisfies $i^2 = -1$. When equality is arranged,

$$GH_{k,n+2} = H_{k,n+2} + iH_{k,n+1}.$$

In Table 3 for definition 6 summarizes some special cases of f(k), g(k), $H_{k,0}$ and $H_{k,1}$.

By giving the $f(k)$, $g(k)$, $H_{k,0}$ and $H_{k,1}$ values, the number sequences in the literature can be reached.

2. GENERALİZED COMPLEX *K*-HORADAM AND GENERALİZED GAUSSİAN *K*-HORADAM RELATED İDENTİTİES

In this section, we have considered generalized complex *k*-Horadam and generalized Gaussian *k*-Horadam sequences. We have given the Binet formulas for these sequences. Then we have founded the generating functions and we obtain some identities involving these sequences. Binet formulas are well-known formulas in the theory number sequences. In the next theorem, we will give the Binet formulas for generalized complex *k*-Horadam and generalized Gaussian *k*-Horadam numbers.

Table 2. Some special cases of $f(k)$, $g(k)$, $H_{k,0}$ and $H_{k,1}$

$[f(k), g(k)]$	$(H_{k,0}, H_{k,1})$	**Complex *k*-Horadam Numbers**
[1, 1]	(0, 1)	Complex Fibonacci Numbers
[2, 1]	(2, 2)	Complex Pell-Lucas Numbers
[1, 2]	(2, 1)	Complex Jacobsthal-Lucas Numbers
[*k*, 1]	(0, 1)	Complex *k*-Fibonacci Numbers

Table 3. Some special cases of $f(k)$, $g(k)$, $H_{k,0}$ and $H_{k,1}$

$[f(k), g(k)]$	$(H_{k,0}, H_{k,1})$	**Gaussian *k*-Horadam Numbers**
[1, 1]	(0, 1)	Gaussian Fibonacci Numbers
[2, 1]	(2, 2)	Gaussian Pell-Lucas Numbers
[1, 2]	(2, 1)	Gaussian Jacobsthal-Lucas Numbers
[*k*, 1]	(0, 1)	Gaussian *k*-Fibonacci Numbers

Theorem 1. Let r_1 and r_2 be the roots of a second order difference equation of generalized *k*-Horadam sequence with recurrence relation $H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n}$. For $H_{k,0} = a$ and $H_{k,1} = b,$ Binet formulas for generalized complex *k*-Horadam and generalized Gaussian *k*-Horadam sequences are given by

$$CH_{k,n} = \frac{(H_{k,1} - r_2H_{k,0})r_1^n - (H_{k,1} - r_1H_{k,0})r_2^n}{r_1 - r_2} + i\frac{(H_{k,1} - r_2H_{k,0})r_1^{n+1} - (H_{k,1} - r_1H_{k,0})r_2^{n+1}}{r_1 - r_2}$$

and

$$GH_{k,n}=\frac{(H_{k,1}-r_2H_{k,0})r_1^n-(H_{k,1}-r_1H_{k,0})r_2^n}{r_1-r_2}+i\,\frac{(H_{k,1}-r_2H_{k,0})r_1^{n-1}-(H_{k,1}-r_1H_{k,0})r_2^{n-1}}{r_1-r_2}$$

respectively.

Proof. When Definition3 is substituted $CH_{k,n}=H_{k,n}+iH_{k,n+1}$ for equation, we get

$$CH_{k,n}=\frac{(H_{k,1}-r_2H_{k,0})r_1^n-(H_{k,1}-r_1H_{k,0})r_2^n}{r_1-r_2}+i\,\frac{(H_{k,1}-r_2H_{k,0})r_1^{n+1}-(H_{k,1}-r_1H_{k,0})r_2^{n+1}}{r_1-r_2}.$$

Similarly, the Binet formula of generalized Gaussian *k*-Horadam sequence

$$GH_{k,n}=\frac{(H_{k,1}-r_2H_{k,0})r_1^n-(H_{k,1}-r_1H_{k,0})r_2^n}{r_1-r_2}+i\,\frac{(H_{k,1}-r_2H_{k,0})r_1^{n-1}-(H_{k,1}-r_1H_{k,0})r_2^{n-1}}{r_1-r_2}$$

can be obtained.

In Table 4 for Binet formula summarizes some special cases of $H_{k,0}$ and $H_{k,1}$:

Table 4. For Binet formula of complex *k*-Horadam sequences, some special cases of $H_{k,0}$ and $H_{k,1}$

Binet Formula	$[f(k), g(k)]$	$(H_{k,0}, H_{k,1})$	Complex *k*-Horadam Numbers
$\frac{r_1^n-r_2^n}{r_1-r_2}+i\,\frac{r_1^{n+1}-r_2^{n+1}}{r_1-r_2}$	[1, 1]	(0, 1)	Complex Fibonacci Numbers
$\frac{(1-2r_2)r_1^n-(1-2r_1)r_2^n}{r_1-r_2}+i\,\frac{(1-2r_2)r_1^{n+1}-(1-2r_1)r_2^{n+1}}{r_1-r_2}$	[1, 1]	(2, 1)	Complex Lucas Numbers
$\frac{(2-2r_2)r_1^n-(2-2r_1)r_2^n}{r_1-r_2}+i\,\frac{(2-2r_2)r_1^{n+1}-(2-2r_1)r_2^{n+1}}{r_1-r_2}$	[2, 1]	(2, 2)	Complex Pell-Lucas Numbers
$\frac{r_1^n-r_2^n}{r_1-r_2}+i\,\frac{r_1^{n+1}-r_2^{n+1}}{r_1-r_2}$	[1, 2]	(0, 1)	Complex Jacobsthal Numbers

By giving the $H_{k,0}$ and $H_{k,1}$ values, the number sequences in the literature can be reached.

In Table 5 for Binet formula summarizes some special cases of $H_{k,0}$ and $H_{k,1}$.

By giving the $H_{k,0}$ and $H_{k,1}$ values, the number sequences in the literature can be reached.

Let r_1 and r_2 be the roots of a second order difference equation of generalized k-Horadam sequence with recurrence relation $H_{k,n+2}=f(k)H_{k,n+1}+g(k)H_{k,n}$. Binet formulas for Gaussian Pell sequence

$$GP_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}+i\,\frac{\alpha\beta^n-\beta\alpha^n}{\alpha-\beta}\quad n\geq 0$$

and Gaussian Pell-Lucas sequence

$$GP_n=(\alpha^n+\beta^n)-i(\alpha\beta^n+\beta\alpha^n)\quad n\geq 0$$

were found by [15]. A different way has been found for the Binet formula of the number sequences by theorem1.

Theorem 2. Let r_1 and r_2 be the roots of a second order difference equation of generalized *k*-Horadam sequence with recurrence relation $H_{k,n+2}=f(k)H_{k,n+1}+g(k)H_{k,n}$. Then, for $H_{k,0}=a$ and $H_{k,1}=b$, we have

$$CH_{k,n}=[r_1H_{k,n-1}+(H_{k,1}-r_1H_{k,0})r_2^{n-1}]+i[r_1H_{k,n}+(H_{k,1}-r_1H_{k,0})r_2^n]$$

and

$$GH_{k,n} = [r_1 H_{k,n-1} + (H_{k,1} - r_1 H_{k,0}) r_2^{n-1}] + i[r_1 H_{k,n-2} + (H_{k,1} - r_1 H_{k,0}) r_2^{n-2}]\,.$$

Proof. When Definition2 is substituted $CH_{k,n} = H_{k,n} + iH_{k,n+1}$ for equation, we get

$$CH_{k,n} = [r_1 H_{k,n-1} + (H_{k,1} - r_1 H_{k,0}) r_2^{n-1}] + i[r_1 H_{k,n} + (H_{k,1} - r_1 H_{k,0}) r_2^{n}]\,.$$

Similarly,

$$GH_{k,n} = [r_1 H_{k,n-1} + (H_{k,1} - r_1 H_{k,0}) r_2^{n-1}] + i[r_1 H_{k,n-2} + (H_{k,1} - r_1 H_{k,0}) r_2^{n-2}]$$

can be obtained.

We can give the generating functions for the generalized complex *k*-Horadam and generalized Gaussian *k*-Horadam sequences in the following theorem.

Table 5. For Binet formula of Gaussian *k*-Horadam sequences, some special cases of $H_{k,0}$ and $H_{k,1}$

Binet Formula	$[f(k), g(k)]$	$(H_{k,0}, H_{k,1})$	Gaussian *k*-Horadam Numbers
$\frac{r_1^n - r_2^n}{r_1 - r_2} + i\frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}$	[1, 1]	(0, 1)	Gaussian Fibonacci Numbers
$\frac{(1-2r_2)r_1^n - (1-2r_1}{r_1 - r_2} + i\frac{(1-2r_2)r_1^{n-1} - (1-2r_1)r_2^{n-1}}{r_1 - r_2}$	[1, 1]	(2, 1)	Gaussian Lucas Numbers
$\frac{(2-2r_2)r_1^n - (2-2r_1)r_2^n}{r_1 - r_2} + i\frac{(2-2r_2)r_1^{n-1} - (2-2r_1)r_2^{n-1}}{r_1 - r_2}$	[2, 1]	(2, 2)	Gaussian Pell-Lucas Numbers
$\frac{r_1^n - r_2^n}{r_1 - r_2} + i\frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}$	[1, 2]	(0, 1)	Gaussian Jacobsthal Numbers

Theorem 3. For $H_{k,0} = a$ and $H_{k,1} = b$, the generating functions for the generalized complex *k*-Horadam and generalized Gaussian *k*-Horadam sequences are

$$CH(x) = \frac{(H_{k,0} + xH_{k,1} - xf(k)H_{k,0}) + i(H_{k,1} + xH_{k,2} - xf(k)H_{k,1})}{1 - xf(k) - x^2 g(k)}$$

and

$$GH_{k,n} = \frac{(H_{k,0} + xH_{k,1} - xf(k)H_{k,0}) + i(H_{k,-1} + xH_{k,0} - xf(k)H_{k,-1})}{1 - xf(k) - x^2 g(k)}$$

respectively.

Proof. Let $CH(x)$ be the generating function of sequence $\{CH_{k,n}\}_{n\in N}$. Then we can write

$$CH(x) = \sum_{n=0}^{\propto} CH_n x^n = CH_0 x^0 + CH_1 x^1 + CH_2 x^2 + \cdots + CH_n x^n + \cdots$$

If we use the recursive relation of this sequence, we get

$$CH(x) = \frac{(H_{k,0} + xH_{k,1} - xf(k)H_{k,0}) + i(H_{k,1} + xH_{k,2} - xf(k)H_{k,1})}{1 - xf(k) - x^2 g(k)}$$

which is desired. Similarly, the generating function of sequence $\{CH_{k,n}\}_{n\in N}$

$$GH_{k,n} = \frac{(H_{k,0} + xH_{k,1} - xf(k)H_{k,0}) + i(H_{k,-1} + xH_{k,0} - xf(k)H_{k,-1})}{1 - xf(k) - x^2g(k)}$$

can be obtained.

In Table 6 for generating function summarizes some special cases of $H_{k,0}$ and $H_{k,1}$:

Table 6. For generating function of complex/gaussian *k*-Horadam sequences, some special cases of $H_{k,0}$ and $H_{k,1}$

Generating Function	$[f(k), g(k)]$	$(H_{k,0}, H_{k,1})$	Complex/Gaussian *k*-Horadam Numbers
$\frac{x+i}{1-x-x^2}$	[1, 1]	(0, 1)	Complex Fibonacci Numbers
$\frac{(2-x)+i(1+2x)}{1-x-x^2}$	[1, 1]	(2, 1)	complex Lucas Numbers
$\frac{x+i(1-2x)}{1-2x-x^2}$	[2, 1]	(0, 1)	Gaussian Pell Numbers
$\frac{(2-2x)+i(6x-2)}{1-2x-x^2}$	[2, 1]	(2, 2)	Gaussian Pell-Lucas Numbers

By giving the $f(x)$, $g(x)$, $H_{k,0}$ and $H_{k,1}$ values, the number sequences in the literature can be reached.

For $f(k) = 2$, $g(k) = 1$, $H_{k,0} = 0$ and $H_{k,1} = 1$, generating function of Gaussian Pell sequence

$$GP(x) = \frac{x + i(1-2x)}{1 - 2x - x^2}$$

and for $f(k) = 2$, $g(k) = 1$, $H_{k,0} = 2$ and $H_{k,1} = 2$, generating function of Gaussian Pell-Lucas sequence

$$GQ(x) = \frac{(2-2x) + i(6x-2)}{1 - 2x - x^2}$$

were found by [15]. A different way has been found for the generating function of the number sequences by theorem 3.

Theorem 4. (*d'Ocagne Formula*) For $H_{k,0} = a$ and $H_{k,1} = b$, the generalized complex *k*-Horadam and generalized Gaussian *k*-Horadam sequences, we have the following equations

$$CH_{k,m+1}CH_{k,n} - CH_{k,m}CH_{k,n+1} =$$

$$\frac{(b^2 - aH_{k,2})\left([-g(k)]^n(bH_{k,n-m} - aH_{k,n-m+1}) + [-g(k)]^{n+1}(bH_{k,m-n} - aH_{k,m-n+1})\right)}{b^2 - a^2g(k) - abf(k)}$$

$$+i\frac{\left([-g(k)]^m(bH_{k,2} - aH_{k,3})(bH_{k,n-m} - aH_{k,n-m+1})\right)}{b^2 - a^2g(k) - abf(k)}$$

and

$$GH_{k,m+1}GH_{k,n} - GH_{k,m}GH_{k,n+1} =$$

$$\frac{(b^2 - aH_{k,2})\left([-g(k)]^{n-1}(bH_{k,m-n} - aH_{k,m-n+1}) + [-g(k)]^{m}(bH_{k,n-m} - aH_{k,n-m+1})\right)}{b^2 - a^2g(k) - abf(k)}$$
$$+i\frac{\left([-g(k)]^{m-1}(bH_{k,2} - aH_{k,3})(bH_{k,n-m} - aH_{k,n-m+1})\right)}{b^2 - a^2g(k) - abf(k)}$$

respectively.

Proof.

$$CH_{k,m+1}CH_{k,n} - CH_{k,m}CH_{k,n+1} =$$

$$(H_{k,m+1} + iH_{k,m+2})(H_{k,n} + iH_{k,n}) - (H_{k,m} + iH_{k,m+1})(H_{k,n+1} + iH_{k,n+2})$$

$$CH_{k,m+1}CH_{k,n} - CH_{k,m}CH_{k,n+1} =$$

$$(H_{k,n} + H_{k,m+1} - H_{k,n+1} + H_{k,m}) + (H_{k,n+2} + H_{k,m+1} - H_{k,n+1} + H_{k,m+2})$$
$$+i(H_{k,n} + H_{k,m+2} - H_{k,n+2} + H_{k,m})$$

By using Definition4, we get

$$CH_{k,m+1}CH_{k,n} - CH_{k,m}CH_{k,n+1} =$$

$$\frac{(b^2 - aH_{k,2})\left([-g(k)]^{n}(bH_{k,n-m} - aH_{k,n-m+1}) + [-g(k)]^{n+1}(bH_{k,m-n} - aH_{k,m-n+1})\right)}{b^2 - a^2g(k) - abf(k)}$$

$$+i\frac{\left([-g(k)]^{m}(bH_{k,2} - aH_{k,3})(bH_{k,n-m} - aH_{k,n-m+1})\right)}{b^2 - a^2g(k) - abf(k)}$$

Similarly, we can prove the other formula by (1.3) equation

$$GH_{k,m+1}GH_{k,n} - GH_{k,m}GH_{k,n+1} =$$

$$\frac{(b^2 - aH_{k,2})\left([-g(k)]^{n-1}(bH_{k,m-n} - aH_{k,m-n+1}) + [-g(k)]^{m}(bH_{k,n-m} - aH_{k,n-m+1})\right)}{b^2 - a^2g(k) - abf(k)}$$

$$+i\frac{\left([-g(k)]^{m-1}(bH_{k,2} - aH_{k,3})(bH_{k,n-m} - aH_{k,n-m+1})\right)}{b^2 - a^2g(k) - abf(k)}$$

Thus, the proof is completed.

In Table 7 for d'Ocagne formula summarizes some special cases of $H_{k,0}$ and $H_{k,1}$.

Table 7. For d'Ocagne formula of complex/gaussian *k*-Horadam sequences, some special cases of $H_{k,0}$ and $H_{k,1}$

d'Ocagne Formula	$[f(k), g(k)]$ and $(H_{k,0}, H_{k,1})$	Complex/Gaussian *k*-Horadam Numbers
Gaussian Lucas Numbers for [1, 1] and (2, 1):		
$[(-1)^m(L_{n-m} - 2L_{n-m+1}) + (-1)^{n-1}(L_{m-n} - 2L_{m-n+1})] + i[(-1)^{m-1}(L_{n-m} - 2L_{n-m+1})]$.		
Complex Pell Numbers for [2, 1] and (0, 1):		
$[(-1)^n(P_{n-m} - P_{m-n})] + i[2(-1)^m P_{n-m}]$.		

Complex Jacobsthal Numbers for [1, 2] and (0, 1):

$[(-2)^n(J_{n-m}-2J_{m-n})]+i[(-2)^m J_{n-m}]$.

Gaussian Fibonacci Numbers for [1, 1] and (0, 1):

$[(-1)^m F_{n-m}+(-1)^{n-1}F_{m-n}]+i[(-1)^{m-1}F_{n-m}]$.

By giving the $f(x)$, $g(x)$, $H_{k,0}$ and $H_{k,1}$ values, the number sequences in the literature can be reached.

d'Ocagne formulas for Gaussian Pell sequence

$$GP_{m+1}GP_n - GP_m GP_{n+1} = 2(-1)^{n+1}(1-i)P_{m-n} \quad n,m \in Z$$

and Gaussian Pell-Lucas sequence

$$GQ_{m+1}GQ_n - GQ_m GQ_{n+1} = 16(-1)^n(1-i)P_{m-n} \quad n,m \in Z$$

were found by [15]. A different way has been found for the d'Ocagne formula of the number sequences by theorem 4.

Theorem 5. (*Catalan Formula*) For $H_{k,0}=a$ and $H_{k,1}=b$, the generalized complex *k*-Horadam and generalized Gaussian *k*-Horadam sequences, we have the following equations

$$CH_{k,m+n}CH_{k,n-m} - CH_{k,n}^2 =$$

$$\frac{(bH_{k,m}-aH_{k,m+1})\left([-g(k)]^n(bH_{k,-m}-aH_{k,-m+1})+[-g(k)]^{n-m+1}(bH_{k,m}-aH_{k,m+1})\right)}{b^2-a^2g(k)-abf(k)}$$

$$+i\frac{(bH_{k,m}-aH_{k,m+1})\left([-g(k)]^n(bH_{k,-m+1}-aH_{k,-m+2})+[-g(k)]^{n+1}(bH_{k,-m-1}-aH_{k,-m})\right)}{b^2-a^2g(k)-abf(k)}$$

and

$$GH_{k,m+n}GH_{k,n-m} - GH_{k,n}^2 =$$

$$\frac{(bH_{k,m}-aH_{k,m+1})\left([-g(k)]^n(bH_{k,-m}-aH_{k,-m+1})+[-g(k)]^{n-m-1}(bH_{k,m}-aH_{k,m+1})\right)}{b^2-a^2g(k)-abf(k)}$$

$$+i\frac{(bH_{k,m}-aH_{k,m+1})\left([-g(k)]^n(bH_{k,-m-1}-aH_{k,-m})+[-g(k)]^{n-1}(bH_{k,-m+1}-aH_{k,-m+2})\right)}{b^2-a^2g(k)-abf(k)}$$

respectively.

Proof.

$$GH_{k,m+n}GH_{k,n-m} - GH_{k,n}^2 =$$

$$(H_{k,m+n}+iH_{k,m+n+1})(H_{k,n-m}+iH_{k,n-m+1})-(H_{k,n}+iH_{k,n+1})(H_{k,n}+iH_{k,n+1})$$

$$GH_{k,m+n}GH_{k,n-m} - GH_{k,n}^2 =$$

$$(H_{k,n+m}+H_{k,n-m}-H_{k,n}+H_{k,n})+(H_{k,n+1}+H_{k,n+1}-H_{k,n+m+1}+H_{k,n-m+1})$$
$$+i[(H_{k,n+m}+H_{k,n-m+1}-H_{k,n}+H_{k,n+1})+(H_{k,n-m}+H_{k,m+n+1}-H_{k,n}+H_{k,n+1})]$$

By using Definition4, we get

$$CH_{k,m+n}CH_{k,n-m} - CH_{k,n}^2 =$$

$$\frac{\left(bH_{k,m} - aH_{k,m+1}\right)\left([-g(k)]^n\left(bH_{k,-m} - aH_{k,-m+1}\right) + [-g(k)]^{n-m+1}\left(bH_{k,m} - aH_{k,m+1}\right)\right)}{b^2 - a^2g(k) - abf(k)}$$

$$+i\frac{\left(bH_{k,m} - aH_{k,m+1}\right)\left([-g(k)]^n\left(bH_{k,-m+1} - aH_{k,-m+2}\right) + [-g(k)]^{n+1}\left(bH_{k,-m-1} - aH_{k,-m}\right)\right)}{b^2 - a^2g(k) - abf(k)}$$

Similarly, we can prove the other formula by (1.3) equation

$$GH_{k,m+n}GH_{k,n-m} - GH_{k,n}^2 =$$

$$\frac{\left(bH_{k,m} - aH_{k,m+1}\right)\left([-g(k)]^n\left(bH_{k,-m} - aH_{k,-m+1}\right) + [-g(k)]^{n-m-1}\left(bH_{k,m} - aH_{k,m+1}\right)\right)}{b^2 - a^2g(k) - abf(k)}$$

$$+i\frac{\left(bH_{k,m} - aH_{k,m+1}\right)\left([-g(k)]^n\left(bH_{k,-m-1} - aH_{k,-m}\right) + [-g(k)]^{n-1}\left(bH_{k,-m+1} - aH_{k,-m+2}\right)\right)}{b^2 - a^2g(k) - abf(k)}$$

Thus, the proof is completed.

In Table 8 for Catalan formula summarizes some special cases of $H_{k,0}$ and $H_{k,1}$:

Table 8. For Catalan formula of complex/gaussian *k*-Horadam sequences, some special cases of $H_{k,0}$ and $H_{k,1}$

Catalan Formula	$[f(k), g(k)]$ and $(H_{k,0}, H_{k,1})$	Complex/Gaussian *k*-Horadam Numbers
Gaussian Lucas Numbers for [1, 1] and (2, 1):		
$\frac{(L_m-2L_{m+1})[(-1)^n(L_{-m}-2L_{-m+1})+(-1)^{n-m+1}(L_m-2L_{m+1})]}{-3} + i\frac{(-1)^n(L_m-2L_{m+1})(L_{-m-1}-2L_{-m}-L_{-m+1}-2L_{-m+2})}{-3}$		
Gaussian Fibonacci Numbers for [1, 1] and (0, 1):		
$F_m((-1)^nF_{-m} + (-1)^{n-m-1}F_m) + i(-1)^nF_m(F_{-m-1} - F_{-m+1})$		
Complex Jacobsthal-Lucas Numbers for [1, 2] and (2, 1):		
$\frac{(R_m-2R_{m+1})[(-2)^n(R_{-m}-2R_{-m+1})+(-2)^{n-m+1}(R_m-2R_{m+1})]}{-9} + i\frac{(-2)^n(R_m-2R_{m+1})(R_{-m+1}-2R_{-m+2}-R_{-m-1}-4R_{-m})}{-9}$		
Complex Pell-Lucas Numbers for [2, 1] and (2, 2):		
$\frac{(Q_m-Q_{m+1})[(-1)^n(Q_{-m}-Q_{-m+1})+(-1)^{n-m+1}(Q_m-Q_{m+1})]}{-2} + i\frac{(-1)^n(Q_m-Q_{m+1})(Q_{-m+1}-Q_{-m+2}-Q_{-m-1}-Q_{-m})}{-2}$		

By giving the $f(x)$, $g(x)$, $H_{k,0}$ and $H_{k,1}$ values, the number sequences in the literature can be reached.

Let r_1 and r_2 be the roots of a second order difference equation of generalized *k*-Horadam sequence with recurrence relation $H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n}$.

Catalan formulas for Gaussian Pell sequence

$$GP_{n+k}GP_{n-k} - GP_n^2 = (-1)^n(1-i)\left[1 + \frac{(-1)^{k+1}\left(r_1^k + r_2^k\right)^2}{4}\right] \quad n,k \in Z^+$$

and Gaussian Pell-Lucas sequence

$$GQ_{n+k}GQ_{n-k} - GQ_n^2 = 2(-1)^{n+1}(1-i)\left[4+(-1)^{k+1}\left(r_1^k + r_2^k\right)^2\right] \quad n,m \in Z^+$$

were found by [15]. A different way has been found for the Catalan formula of the number sequences by theorem5.

Theorem 6. (*Cassini Formula*) For $H_{k,0} = a$ and $H_{k,1} = b$, the generalized complex *k*-Horadam and generalized Gaussian *k*-Horadam sequences, we have the following equations

$$CH_{k,n+1}CH_{k,n-1} - CH_{k,n}^2 =$$

$$\frac{\left(b^2 - aH_{k,2}\right)\left(\left[-g(k)\right]^n\left(bH_{k,-1} - a^2\right) + \left(b^2 - aH_{k,2}\right)\right)}{b^2 - a^2 g(k) - abf(k)}$$

$$+i\frac{\left(b^2 - aH_{k,2}\right)\left[-g(k)\right]^{n+1}\left(bH_{k,-2} - aH_{k,-1}\right)}{b^2 - a^2 g(k) - abf(k)}$$

and

$$GH_{k,n+1}GH_{k,n-1} - GH_{k,n}^2 =$$

$$\frac{\left(b^2 - aH_{k,2}\right)\left(\left[-g(k)\right]^n\left(bH_{k,-1} - a^2\right) + \left[-g(k)\right]^{n-2}\left(b^2 - aH_{k,2}\right)\right)}{b^2 - a^2 g(k) - abf(k)}$$

$$+i\frac{\left(b^2 - aH_{k,2}\right)\left[-g(k)\right]^{n}\left(bH_{k,-2} - aH_{k,-1}\right)}{b^2 - a^2 g(k) - abf(k)}$$

respectively.

Proof. If $m = 1$ is taken in the Catalan formula, Cassini formula is obtained.

In Table 9 for Cassini formula summarizes some special cases of $H_{k,0}$ and $H_{k,1}$.

By giving the $f(x), g(x)$, $H_{k,0}$ and $H_{k,1}$ values, the number sequences in the literature can be reached.

Table 9. For Cassini formula of Complex/Gaussian *k*-Horadam sequences, some special cases of $H_{k,0}$ and $H_{k,1}$

Cassini Formula	$[f(k), g(k)]$	$(H_{k,0}, H_{k,1})$	Complex/Gaussian *k*-Horadam Numbers
$[(-1)^n + 1] - i2(-1)^{n+1}$	[2, 1]	(0, 1)	Complex Pell Numbers
$\left[\frac{(-9)(-2)^n - 18}{2}\right] + i\left[\frac{9(-2)^{n+1}}{4}\right]$	[1, 2]	(2, 1)	Complex Jacobsthal-Lucas Numbers
$[(-10)(-1)^n] + i5(-1)^n$	[1, 1]	(0, 2)	Gaussian Lucas Numbers
$\left[\frac{(-2)^n + 2(-2)^{n-2}}{2}\right] + i\left[\frac{(-1)(-2)^n}{4}\right]$	[1, 2]	(0, 1)	Gaussian Jacobsthal Numbers

3. CONCLUSIONS

This study has examined and studied complex-type sequences with the help of a simple and general formula. For this purpose, generalized Complex *k*-Horadam and generalized Gaussian *k*-Horadam sequences $\{CH_{k,n}\}_{n\in N}$ and $\{GH_{k,n}\}_{n\in N}$ have been used. In this study, Binet formulas, generating functions and some identities of all Complex *k*-Horadam and Gaussian *k*-Horadam sequences have been obtained. As a result, formulas in the literature have been given with the help of one formula. Number sequences have great importance as they are used in physics, applied mathematics and di□erential equations.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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Peer-review history:
The peer review history for this paper can be accessed here:
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