Asian Research Journal of Mathematics

Volume 19, Issue 1, Page 45-55, 2023; Article no.ARJOM.95831
ISSN: 2456-477X

Locally Attractivity Solution of Coupled Fractional Integral Equation in Banach Algebras

Pravin M. More [a*], B. D. Karande [b] and P. S. Amrutrao [c]

[a] Department of Mathematics, B.S.S. Arts Science and Commerce College, Makani, Maharashtra, India.
[b] Department of Mathematics, Maharashtra Udayagiri Mahavidalaya, Udgir, Maharashtra, India.
[c] Department of Mathematics, B.S.S. Arts Science and Commerce College, Naldurg, Maharashtra, India.

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2023/v19i1637

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/95831

Original Research Article

Received: 27/10/2022
Accepted: 29/12/2022
Published: 28/01/2023

Abstract

In this paper we prove the Locally Attractivity solution of coupled hybrid functional integral equation of fractional order (CQHFIE) in Banach algebras. Our main result is based on the hybrid fixed point theory used theorem due to Dhage B. C.

Keywords: *Couple Quadratic Hybrid Functional Integral Equation (CQHFIE); attractive results; hybrid couple fixed point theory; Lipchitz conditions.*

1 Introduction

"Nonlinear analysis for studying the dynamical systems represent the nonlinear differential and integral equation" [1]. "The solutions of nonlinear differential equations are requires to development of different

**Corresponding author: Email: pravinmopavin@gmail.com, pravinmorepravin@gmail.com;*

techniques in order to deduce the existence and other essential properties of the solutions" [2-4]. "There are still many authors are used open problems related to the solvability of nonlinear systems in fractional order" [5-11]. The Coupled Hybrid fractional integral and differential equation are usually study from the different type of areas of applied sciences, biological sciences, and chemical sciences and various mechanical and fluid dynamics process [12,13,14]. The couple systems study of the existence, uniqueness, stability and other behavior of solutions see Hu [9]. "Nonlinear fractional integral equations have been discussed in the literature long time see B.C.Dhage, D. O'Rogan [15] and there references. Quadratic integral equations are often the applicable in the theory of radioactive transfers, kinetic theory of gases, the queening theory and traffic theory, especially the so-called as quadratic many authors studied the existence of solutions several classes of nonlinear quadratic integral equations" see ([16,17,18,19,14]). "The most above the literature the main result realized with the help of the technique measure of non compactness we use the hybrid fixed point theory was proved Schauder Tynchof fixed point theorem and also existence of some coupled systems of Chandrasekhar integral equations will be applicable" [20,21,22,14].

In this paper is to we prove the Locally Attractivity solution of coupled hybrid functional integral equation of fractional order (CQHFIE) in Banach algebras along with the locally attractivity solutions under the Lipchitz conditions.

Let $\alpha \epsilon (0,1)$ and R denote the real numbers where as R_+ be the set of nonnegative numbers ie, $R_+ = [0, \infty) \subset R$.

Consider the coupled system of quadratic hybrid functional integral equation (CQHFIE).

$$\left.\begin{aligned} x(t) &= \frac{f\big(t, y(\mu(t))\big)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g\left(s, y\big(\gamma(s)\big)\right) ds \ \forall\ t \epsilon R_+ \\ y(t) &= \frac{f\big(t, x(\mu(t))\big)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g\left(s, x\big(\gamma(s)\big)\right) ds \ \forall\ t \epsilon R_+ \end{aligned}\right\} \tag{1.1}$$

Where $\alpha \epsilon (0,1)$and function.

$f(t,x) = f: R_+ \times R \to R$, $g(t,x) = g: R_+ \times R \to R$, $\mu, \gamma: R_+ \to R_+$ are continuous .

By a solution of system of CQHFIE (1.1) we mean the function $(x,y) \epsilon BC\big(R_{+,} R \times R\big)$ is the space of continuous and bounded real valued function defined on $R_{+.}$

Firstly, we convert CQHFIE (1.1) into an operator equation then we apply the coupled fixed point theorem [6]. in Banach algebras under some suitable conditions on the nonlinearities $fandg$.

2 Preliminaries

Definition 2.1[23] (Dugundji and Granass): an operator A on Banach space X into itself is called compact if for any bounded subset S of X, $A(S)$ is relatively compact subset of X. If A is continuous and compact then it is called completely continuous on X.

Let X be a Banach space with norm $\|.\|$ and $A: X \to X$ be an operator (in general nonlinear) then A is called,

i) Compact if $A(X)$ is relatively compact subset of X.
ii) Totally bounded if $A(S)$is totally bounded subset of X for any bounded subset S of X.
iii) Completely continuous if it is continuous and totally bounded operator on X. It is clear that every compact operator is totally bounded but not converse need not true.
Let $X = BC(R_+, R)$ be algebra of continuous real valued functions defined on R_+.Define a norm $\|.\|$and multiplication in X by,

$$\|x\| = \sup_{t \epsilon R_+} |x(t)| \text{ and } \quad (x,y)(t) = x(t), y(t) \quad \forall t \epsilon R_+ \tag{2.1}$$

Clearly X is Banach space with respects to above norm and multiplication in it. By $\|.\|_{L^1}$ in $BC\big(R_{+,} R\big)$ by

$$\|x\|_{L^1} = \int_0^T |x(s)|\ ds \tag{2.2}$$

Denote by $L_1(a,b)$ be the Banach space of the Lebesguge integrable functions on then interval (a,b) which equipped with standard norm .Let $x\epsilon L_1(a,b)$and let $\beta > 0$ be a fixed number.

Definition 2.2 [23]:.Let $X = BC(R_+, R)$ be the Banach algebra with norm $\|.\|$ and let Ω be a subset of X. Let's a mapping $A: X \to X$ be an operator and consider the following operator equation in X namely,

$$x(t) = (Ax)(t)\ \ t\epsilon R_+ \tag{2.3}$$

Definition 2.3 [7,8]: An element $(x,y)\epsilon X \times X$ is called coupled fixed point of mapping $T: X \times X \to X$ if $T(x,y) = x$ and$T(y,x) = y$.

Definition 2.4 [25]: The solution $x(t)$ of the equation CQHFIE (1.1) is said to be locally attractive if there exists an closed ball $B_r[0]$ in $BC(R_+, R)$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (1.1) belonging to $B_r[0] \cap \Omega$ such that,

$$\lim_{t\to\infty}\big(x(t) - y(t)\big) = 0 \tag{2.4}$$

Definition 2.5 [25]: Let X be the Banach algebra .A mapping $A: X \to X$ is called Lipachtiz if there is constant $\alpha > 0$ such $\|Ax - Ay\| \le \|x - y\|$ for all $x, y\epsilon X$. if $\alpha < 1$ then A is called a contraction constant α.

Definition 2.6 [12,26]: A set $A \subseteq [a,b]$ is said to be measurable if $m^*A = m_*A$. In case we define mA, the measure of A as $mA = m^*A = m_*A$

If A_1 and A_2 are measurable of $[a,b]$ then their union and intersection is also measurable. Clearly every open or closed set in R is measurable.

Definitain 2.7 [12]: Let$f$ be a function defined on [a, b] then f is measurable function if for each $\alpha\epsilon R$, the set $\{x: f(x) > \alpha\}$ is measurable set. i.e. f is measurable function if for every real number $\alpha$of inverse image of (α, ∞) is open set. As (α, ∞) is an open set if f is continuous then inverse image under f of (α, ∞) is open sets being measurable, hence every continuous function is measurable.

Definition 2.8 [15]: A sequence of function $\{f_n\}$ is said converges uniformly interval [a, b] to a function f if for any $\epsilon > 0$ and for all $x\epsilon[a,b]$ then there exist integer N (dependent only on ϵ) such that for all $x\epsilon[a,b]$,

$$|f_n(x) - f(x)| < \epsilon\ \ \forall\ n \ge N$$

Definition 2.9 [12,21]: Let $X = BC(R_+, R)$ equipped with the supermum norm . Clearly it is Banach space with respect to point wise operation and the supremum norm. Define scalar multiplication and sum of $X \times X$as followes,

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2)$ And

$\theta(x,y) = (\theta x, \theta y)$ For $\theta\epsilon R$ then $X \times X$ is vector space on R.

Definition 2.10[17]: Let $\bar{X} = X \times X$ then Define $\|(x,y)\| = \|x\| + \|y\|$ then $\bar{X}$ is Banach space with respect to above norm.

Definition 2.11[17]: An operator $X: E \to E$ is called $\sigma -$ nonlinear contraction if there exist a real constant $\sigma\epsilon(0,1)$ and function $\varphi W\epsilon\phi$ such that $\|w\theta - wv\| = \sigma\varphi w\|\theta - v\|$ for every $\theta, v\ \epsilon\ E$, we call θw a nonlinear function of W on X.

Definition2.11 [26]: The Riemann-Liouville Fractional derivative of order $\alpha > 0$ of a continuous function f:$(0,\infty) \to R$ is given by,

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}\,ds}$$

Where $= [\alpha] + 1$. Let $[\alpha]$ denotes the integer part of number α provided tha the right side point wise on $(0, +\infty)$,

Definition2.12 [6]: The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f: (0 + \infty) \to R$is given by,

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\, ds$$

Provided that right side point wise defined on$(0, +\infty)$.

Theorem2.1 [26,7,8]: Let S be a non-empty, closed convex and bounded subset of the Banach space X and $\bar{S} = S \times S$ and Let $A: X \to X$ and $B: S \to X$ be two operators such that,

a) A is Lipschitzian with Lipchitz constant $\boldsymbol{\alpha}$.
b) B is completely continuous
c) $x = AxBy \Rightarrow x\epsilon S$ for all $y\epsilon S$ and
d) $\alpha M < 1$, where $M = \|B(S)\| = sup\{\|B(x)\|: x\epsilon S\}$

Then the operator equation $T(x, y) = Ax.By$ has at least a coupled point solution in$\bar{S}$.

Theorem 2.2 [Arzela-Ascoli theorem (6)]: Every uniformly bounded and equicontinuous sequence $\{f_n\}$ of function in $BC(R_+, R)$ has convergent subsequence.

Theorem 2.3 [26,24]: A metric space X is compact iff every sequence in X has convergent subsequence.

3 Existence results

Defination3.1 [13]: A mapping $g: R_+ \times R \to R$ is said to be Caratheodory if

1. $t \to g(t, x)$is measurable for all $x \in R$and,
2. $x \to g(t, x)$is continuous almost everywhere for $t\epsilon R_+$again caratheodory function g is called L^1-Carathedory.
3. If for each real number $r > 0$ there exists a function $h_r \epsilon L^1(R_+, R)$such that $|g(t, x)| \le h_r(t)$ a.e.$t\epsilon R_+$for all$x\epsilon R$, with $|x| \le r$, Finally, a Carathedory function $g(t, x)$ called L^1-Carathedory .
4. There exist a function $h\epsilon L^1(R_+, R)$ such that $|g(t, x)| \le h(t)$ i.e. $t\epsilon R_+$ for all $x\epsilon R$ .for convenience, the function h is referred to as bound function of g.

We consider the coupled nonlinear hybrid functional integral equation CQHFIE (1.1) assuming the following hypothesis are satisfied.

$\boldsymbol{H_1}$) $\mu, \gamma: R_+ \to R_+$ are continuous.

$\boldsymbol{H_2}$) The function $f(t, x): R_+ \times R \to R$ is continuous and bounded with bound,

$F = \underbrace{sup}_{(t,x)\epsilon R_+ \times R} |f(t, x)|$ there exist a bounded function $l: R_+ \to R_+$ with bound L satisfying $|f(t, x) - f(t, y)| \le |x(t) - y(t)|$ $\forall t\epsilon R_+$ for all $x, y\epsilon R$.

$\boldsymbol{H_3}$) The function $g: R_+ \times R \to R$satisfy caratheodory condition (i.e. measurable in t for all $x\epsilon R$ and continuous in x for all $t\epsilon R_+$) and there exist function $m_1\epsilon L^1(R_+, R)$ such that $g(t, x) \le h(t)$ $\forall$ $(t, x)\epsilon R_+ \times R$.

$\boldsymbol{H_4}$) The uniform continuous function $v: R_+ \to R_+$ defined formula,

$$v(t) = \int_0^t (t-s)^{\alpha-1} h(s)\, ds$$

Is bounded on R_+ and vanish at infinity, that is $\lim_{t\to\infty} v(t) = 0$.

Remark 3.1: Note that if the hypothesis (H_3) hold, then there exist constant $K_1 > 0$ such that,

$$K_1 = \int_0^t (t-s)^{\alpha-1} h(s)\, ds.$$

4 Main results

Theorem4.1: Suppose that the hypothesis $[H_1 - H_4]$ holds. Further more $\frac{T^\alpha \|h\|_{L^1}}{\Gamma(\alpha+1)} < 1$ .then the equation (1.1) has a solution in the space$BC(R_+, R)$. Moreover, solutions of the equation (1.1) are locally attractive on R_+.

Proof: Let $X = BC(R_+, R)$ be Banach algebras of all continuous and bounded real valued function on R_+ with norm $\|x\| = \underbrace{sup}_{t\epsilon R_+} |x(t)|$.We show that existence solution CQHFIE (1.1) under some suitable conditions on the function involved in (1.1).Set $X = BC(R_+, R)$ is closed subset $B_r[0]$ of X centered at origin O and radius r defined by $B_r[0] = \{x\epsilon X, \|x\| \le r\}$,

$$B_r[0] = \{x\epsilon X\ \|x\| \le r\}, \tag{4.1}$$

Where $[\|F\|]\|h\| \frac{T^\alpha}{\Gamma(\alpha+1)} = r > 0$.

Clearly $S = B_r[0]$ be a nonempty, convex, closed and bounded subset of the Banach spacr X .Define two operators$A: X \to X\ and\ B: B_r[0] \to X$ by,

$$Ax(t) = f\left(t, x(\mu(t))\right) \tag{4.2}$$

$$Bx(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g\left(s, x(\gamma(s))\right)\, ds \tag{4.3}$$

Then the coupled system (1.1) transformed into system of operator equation as,

$$\left.\begin{aligned} x(t) &= Ay(t)By(t) \\ y(t) &= Ax(t)Bx(t) \end{aligned}\right\} \ \forall\ t\epsilon R_+ \tag{4.4}$$

It is sufficient to proves that $A(x,y)\ B(x,y) = (x,y)$ has at least one solution, because,

$$\begin{aligned} \left(T(x,y)T(y,x)\right) &= (AxBy, Ay, Bx) \\ &= (Ax, Ay)(By, Bx) \\ &= A(x,y)B(y,x) \\ &= (x,y) \end{aligned} \tag{4.5}$$

Which implies that $T(x,y)$ has at least one coupled fixed point.

Therefore A, B define the operator $A, B: B_r[0] \to X$. we wish to show that A, B satisfy all the requirement of theorem (2.1) on $B_r[0]$.

Step I: Firstly we show that A is Lipschtiz constant.

Let $x, y \epsilon X$ be arbitrary, and then by hypothesis(H_2) we get,

$$\begin{aligned} |Ax(t) - Ay(t)| &= \left| f\left(t, x(\mu(t))\right) - f\left(t, y(\gamma(t))\right) \right| \\ &= L|x(t) - y(t)| \\ &= \boldsymbol{L}\|x - y\| \text{ for all } t \epsilon R_+ \end{aligned} \tag{4.6}$$

Taking supremum over t.

$\|Ax - Ay\| \leq L\|x - y\|$ for all $x, y \epsilon X$ with Lispschtiz constant L.

Step II: Secondly, to prove that the operator B is completely continuous operator (B is compact and continuous operator) on $B_r[0]$.

Case I: Firstly we show that B is continuous on $B_r[0]$ .Let by dominated convergence theorem, Let $\{x_n\}$ be a sequence in S such that $\{x_n\} \to x$ then,

$$\begin{aligned} \lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g\left(s, x_n(s)\right) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lim_{n\to\infty} g\left(s, x_n(s)\right) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, g\left(s, x_n(s)\right) ds \\ &= Bx(t)\,,\ \forall\, t \epsilon R_+. \end{aligned}$$

This shows that Bx_nconvergence to Bx point wise on S.

Next to show that the sequence $\{Bx_n\}$ is equicontinuous sequence in S.

Let $t_{1,} t_2 \epsilon R_+$ be arbitrary with $t_1 < t_2$ then,

$$\begin{aligned} |Bx_n(t_2) - Bx_n(t_1)| &= \left| \begin{matrix} \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} g\left(s, x_n(s)\right) ds \\ -\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} g\left(s, x_n(s)\right) ds \end{matrix} \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2 - s)^{\alpha-1} g\left(s, x_n(s)\right) ds - \int_0^{t_2} (t_1 - s)^{\alpha-1} g\left(s, x_n(s)\right) ds \right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_1 - s)^{\alpha-1} g\left(s, x_n(s)\right) ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} g\left(s, x_n(s)\right) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2 - s)^{\alpha-1} h(s) ds - \int_0^{t_2} (t_1 - s)^{\alpha-1} h(s) ds \right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_1 - s)^{\alpha-1} h(s)\, ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} h(s) ds \right| \\ &\leq \frac{\|h\|_{L^1}}{\Gamma(\alpha)} \left\{ \left| \int_0^{t_2} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] ds \right| + \left| \int_{t_1}^{t_2} (t_1 - s)^{\alpha-1} ds \right| \right\} \\ &\leq \frac{\|h\|_{L^1}}{\Gamma(\alpha)} \left\{ \left| \left[\frac{(t_2 - s)^{\alpha}}{\alpha}\right]_0^{t_2} - \left[\frac{(t_1 - s)^{\alpha}}{\alpha}\right]_0^{t_2} \right| + \left| \left[\frac{(t_1 - s)^{\alpha}}{\alpha}\right]_{t_1}^{t_2} \right| \right\} \\ &\leq \frac{\|h\|_{L^1}}{\alpha\Gamma(\alpha)} \left\{ \begin{matrix} |[(t_2 - t_2)^{\alpha} - (t_2 - 0)^{\alpha}] - [(t_2 - t_1)^{\alpha} - (t_1 - 0)^{\alpha}]| \\ +[(t_1 - t_2)^{\alpha} - (t_1 - t_1)^{\alpha}] \end{matrix} \right\} \\ &\leq \frac{\|h\|_{L^1}}{\Gamma(\alpha+1)} \{|(t_1)^{\alpha} - (t_2)^{\alpha} - (t_1 - t_2)^{\alpha}| + |(t_1 - t_2)^{\alpha}|\} \\ &\to 0 \text{ as } t_1 \to t_2\ \forall\, n \epsilon N. \end{aligned}$$

This shows that, by using property of uniform convergence is convergence implying continuity.

Hence B is continuous on $B_r[0]$.

Case II: To show B is compact on$B_r[0]$, for this to show that B is uniformly bounded and equicontinuous in $B_r[0]$. First we show that B is uniformly bounded. Let $x \epsilon S$ be arbitrary then,

$$
\begin{aligned}
|Bx(t)| &= \left|\int_0^t (t-s)^{\alpha-1} g\left(s, x\left(\gamma(s)\right)\right)\, ds\right| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|g\left(s, x\left(\gamma(s)\right)\right)\right|\, ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds \\
&\le \frac{1}{\Gamma(\alpha)}\, v(t)
\end{aligned}
$$

Taking supremum over t, we obtain,.

$$\|Bx\| \le \frac{v(t)}{\Gamma(\alpha)} = K_1, \forall\, t \epsilon R_{+.}$$

Hence B is uniformly bounded subset of $B_r[0]$.

Now to show B is equicontinuous on $B_r[0]$.

Let $t_1, t_2 \epsilon R_+$ then,

$$
\begin{aligned}
|Bx(t_2) - Bx(t_1)| &= \left|\begin{matrix} \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} g\left(s, x(s)\right) ds \\ -\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} g\left(s, x(s)\right) ds \end{matrix}\right| \\
&\le \frac{1}{\Gamma(\alpha)} \left|\int_0^{t_2} (t_2 - s)^{\alpha-1} g\left(s, x(s)\right) ds - \int_0^{t_2} (t_1 - s)^{\alpha-1} g\left(s, x(s)\right) ds\right| \\
&\quad + \frac{1}{\Gamma(\alpha)} \left|\int_0^{t_2} (t_1 - s)^{\alpha-1} g\left(s, x(s)\right) ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} g\left(s, x(s)\right) ds\right| \\
&\le \frac{1}{\Gamma(\alpha)} \left|\int_0^{t_2} (t_2 - s)^{\alpha-1} h(s) ds - \int_0^{t_2} (t_1 - s)^{\alpha-1} h(s) ds\right| \\
&\quad + \frac{1}{\Gamma(\alpha)} \left|\int_0^{t_2} (t_1 - s)^{\alpha-1} h(s)\, ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} h(s) ds\right| \\
&\le \frac{\|h\|_{L^1}}{\Gamma(\alpha)} \left\{ \left|\int_0^{t_2} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] ds\right| + \left|\int_{t_1}^{t_2} (t_1 - s)^{\alpha-1} ds\right| \right\} \\
&\le \frac{\|h\|_{L^1}}{\Gamma(\alpha)} \left\{ \left|\left[\frac{(t_2 - s)^\alpha}{\alpha}\right]_0^{t_2} - \left[\frac{(t_1 - s)^\alpha}{\alpha}\right]_0^{t_2}\right| + \left|\left[\frac{(t_1 - s)^\alpha}{\alpha}\right]_{t_1}^{t_2}\right| \right\} \\
&\le \frac{\|h\|_{L^1}}{\alpha\Gamma(\alpha)} \left\{ \begin{matrix} |[(t_2 - t_2)^\alpha - (t_2 - 0)^\alpha] - [(t_2 - t_1)^\alpha - (t_1 - 0)^\alpha]| \\ +[(t_1 - t_2)^\alpha - (t_1 - t_1)^\alpha] \end{matrix} \right\} \\
&\le \frac{\|h\|_{L^1}}{\Gamma(\alpha+1)} \{|(t_1)^\alpha - (t_2)^\alpha - (t_1 - t_2)^\alpha| + |(t_1 - t_2)^\alpha|\} \\
&\to 0 \text{ as } t_1 \to t_2.
\end{aligned}
$$

Implies that B is equicontinuous .Hence B is compact subset of $B_r[0]$.

Therefore it follows from Arzela-Ascolii theorem B is completely continuous on $B_r[0]$.

Step III: Next we show that $x(t) = AxBy \epsilon B_r[0]$ is arbitrary then,

$$
\begin{aligned}
|x(t)| &= |Ax(t)Bx(t)| \le |Ax(t)|.\, |By(t)| \\
&\le \left|f\left(t, x\left(\mu(t)\right)\right)\right| \left|\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g\left(s, x\left(\gamma(s)\right)\right)\, ds\right| \\
&\le \left|f\left(t, x\left(\mu(t)\right)\right)\right| \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|g\left(s, x\left(\gamma(s)\right)\right)\right|\, ds\right] \le |F| \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |h(s)\, | ds\right] \\
&\le \|F\| \|h\| \frac{T^\alpha}{\Gamma(\alpha+1)} = r.
\end{aligned}
$$

Taking the superemum over t we obtain $\|AyBy\| \le r$ for all $x, y \epsilon B_r[0]$.

Hence hypothesis (c) of theorem (2.1) holds. Has a solution on $R_{+.}$

Also we have $M = \|B(B_r[0])\| \le sup\{\|Bx\|: x \epsilon B_r[0]\}$

$$= sup\left\{\underbrace{sup}_{t\geq 0}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g\left(s, x\big(\gamma(s)\big)\right) ds\right\} : x\epsilon B_r[0]\right\}$$
$$\leq \sup\left\{\underbrace{sup}_{t\geq 0}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s) ds \;\; x\epsilon B_r\,[0]\right\}\right\}$$
$$\leq \underbrace{sup}_{t\geq 0}\left\{\frac{v(t)}{\Gamma(\alpha)}\right\} \leq K_1$$

We have$LM = (LK_1) < 1$.

Now we applying theorem the operator $T(x, y) = AxBy$ has at least a coupled fixed point, which implies (1.1) has a solution on $R_{+.}$

Step IV: Now to prove that locally attractivity of solution (2.1).Let assume that x and y be any two solution of (2.1) in $B_r[0]$defined on $R_{+.}$then we have,

$$|x(t)-y(t)| = \left|\begin{array}{l} f\left(t, x\big(\mu(t)\big)\right)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g\left(s, x\big(\gamma(s)\big)\right) ds \\ -f\left(t, y\big(\mu(t)\big)\right)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g\left(s, y\big(\gamma(s)\big)\right) ds \end{array}\right|$$
$$\leq \left|f\left(t, x\big(\mu(t)\big)\right)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g\left(s, x\big(\gamma(s)\big)\right) ds\right|$$
$$+ \left|f\left(t, y\big(\mu(t)\big)\right)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g\left(s, y\big(\gamma(s)\big)\right) ds\right|$$
$$\leq \left|f\left(t, x\big(\mu(t)\big)\right)\right| \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \left|g\left(s, x\big(\gamma(s)\big)\right)\right| ds\right|$$
$$+ \left|f\left(t, y\big(\mu(t)\big)\right)\right| \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \left|g\left(s, y\big(\gamma(s)\big)\right)\right| ds\right|$$
$$\leq |F|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s) ds + |F|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\, ds$$
$$|x(t)-y(t)| \leq 2\|F\|\frac{v(t)}{\Gamma(t)}.$$

For all $t\epsilon R_+$. Since and $\lim_{t\to\infty} v(t) = 0$ this gives $\lim_{t\to\infty} sup|x(t) - y(t)| = 0.$ Thus the (1.1) has solution and all the solutions are locally attractive on R_+.

5 Example

Example 5.1: Consider the following coupled system of equation type (1.1)

$$D^{\frac{1}{2}}\left[\frac{x(t)}{cos\left[\frac{x(t)}{1-x(t)} + \log t\right]}\right] = \frac{1}{t\big(x + x(2t)\big)} \qquad \forall\, t \in \mathbb{R}_+$$

$$D^{\frac{1}{2}}\left[\frac{y(t)}{cos\left[\frac{y(t)}{1-y(t)} + \log t\right]}\right] = \frac{1}{t\big(y + y(2t)\big)} \qquad \forall\, t \in \mathbb{R}_+$$

$$f\left(t, x\big(\mu(t)\big)\right) = cos\; t\;\left[\frac{x(t)}{1-x(t)} + \log t\right]$$

$$g\left(s, x\big(\gamma(s)\big)\right) = \frac{1}{t\big(x + x(2t)\big)}$$

Here $\alpha = \frac{1}{2}, \mu = t, \gamma = 2t$ are continuous on $\mathbb{R}_+$.

a) Hypothesis (H_1) satisfied. Obliviously $\alpha = \frac{1}{2}, \mu = t, \gamma = 2t$ are continuous.

b) To prove the hypothesis (H_2) is satisfied.

Let $\left|f\left(t, x(\mu(t))\right) - f\left(t, y(\mu(t))\right)\right|$

$$= \left|cost\left(\frac{x(t)}{1-x(t)} + \log t\right) - cost\left(\frac{y(t)}{1-y(t)} + \log t\right)\right|$$

$$= \left|cost\frac{x(t)}{1-x(t)} + costlogt - cost.\frac{y(t)}{1-y(t)} - costlogt\right|$$

$$= \left|\cos t\left(\frac{x(t)}{1-x(t)} - \frac{y(t)}{1-y(t)}\right)\right|$$

$$= |cost|\left|\frac{x(t) - x(t)y(t) - y(t) + x(t)y(t)}{(1-x(t))(1-y(t))}\right|$$

$$\leq |cost|\,|x(t) - y(t)|$$

$$\leq l(t)|x(t) - y(t)|$$

$\therefore$ $l(t) = cost$ which has bound $L = 1$ on $\mathbb{R}_+$.

Hence hypothesis (H_2) is satisfied.

c) To show that hypothesis (H_3) is satisfied.

$$\left|g\left(t, x(\gamma(t))\right)\right| = \left|\frac{1}{t(x + x(2t))}\right| \leq \frac{1}{t}$$

Here we take $h(t) = \frac{1}{t}$ it is continuous on $\mathbb{R}_+$

$$\therefore \quad \left|g\left(t, x(\gamma(t))\right)\right| \leq h(t).$$

Hence Hypothesis (H_3) is satisfied with bound $h(t) = \frac{1}{t}$.

d) To show that hypothesis (H_4) is satisfied.

$$v(t) = \int_0^t (t-s)^{\frac{1}{2}} h(t)ds$$

$$= \int_0^t (t-s)^{\frac{1}{2}-1}.\frac{1}{t}\, ds$$

$$= \frac{1}{t}\int_0^t \left[\frac{(t-s)^{\frac{1}{2}}}{-\frac{1}{2}}\right]_0^t$$

$$= \frac{1}{t}\left[0 - \frac{(t-0)^{\frac{1}{2}}}{-\frac{1}{2}}\right]$$

$$= \frac{1}{t}\left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right] = \frac{t^{\frac{1}{2}}}{t}\,\frac{1}{\frac{1}{2}} = \frac{2t^{\frac{1}{2}}}{t^{\frac{1}{2}}.t^{\frac{1}{2}}} = \frac{2}{t^{\frac{1}{2}}}$$

$$\therefore\ v(t) = \frac{2}{t^{\frac{1}{2}}}.$$

Implies that $v(t)$ is bounded for $t \in \mathbb{R}_+$. Hence the hypothesis are satisfied. Consequently all the conditions of theorem (6.2.1) are satisfied. Thus the problem has at least one solution on $\mathbb{R}_+$.

6 Conclusion

In This Paper we have studied the attractivity results of coupled quadratic hybrid functional integral equations of fractional order. The result can be obtained by using hybrid fixed point theorem due to B.C.Dhage in Banach Algebras.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Ahmed El-sayed, Hind Hashem: existence results for nonlinear quadratic integral equations of fractional order in Banach algebra; Fractional Calculus and Applied Analysis. 16(4):816-826.

[2] Kailbas AA, Srivastava HH, Trujillo JJ. Theory and applications of fractional differential equations Elsevier Science B. V. Amsterdam; 2006.

[3] Ahemad B, Ntouyas SK, Alsaedi AA. Existence results for a system of coupled hybrid fractional differential equations The Scientific World Journal; 2014:1-6.

[4] Dhage BC, partially condensing mapping in partially ordered normed linear space and application to functional integral equation Tamkag Journal Mathematics. 2014;45:397-426.

[5] Dhage BC. On fixed point theorem in Banach algebra with application appl. Math. Lett. 2005;18:273-280.

[6] Oldham KB. J. Spanier; The fractional Calculus, Academic Press New York; 1974.

[7] Thereh Bashiri, Sayed Mansor Vaezpour and Choonkil park; A coupled fixed point theorem and application to fractional hybrid differential problems; Fixed Point Theory and Applications. 2016:23.

[8] Thereh Bashiri, sayed Mansor Vaezpour and Choonkil Park; Existence results for hybrid differential systems in Banach algebras, Springer, Advance in Difference Equations; 2016.

[9] Hu X, Yan J. The global attractively and asymptotic stability of solution of nonlinear integral equation J math anal Appl. 2006;321:147-156.

[10] Su X. Boundary value problem for coupled system of nonlinear differential equations applied mathematics Letters. 2009;22:64-69.

[11] Chen YH. An Numerical solutions of coupled Burgers equations with time and space fractional derivatives Appl. Math. Comput. 2008;200:87-95.

[12] Dhage BC, Dhage SB. Coupled hybrid fixed point theorem in partially ordered metric space with applications, Nonlinear stud. 2014;21(4):675-686.

[13] Karande BD, Yachwad SS. Fractional hybrid functional integral equations and a coupled fixed point Theorem, online International Journal. 2017;07:17-28.

[14] Chang SS, Cho YJ, Hung NJ. Coupled fixed point theorems with applications Journal of Korean Mathematical Society. 1996;33:575-585.

[15] Dhage BC. D. O'Regan. : A fixed point theorem in Banach algebras with applications to functional integral equations Function, Differ. Equ. 200;7:259-567.

[16] Dhage BC. Periodic boundary value problems of first order caratheodory and discontinuous differential equations, Nonlinear functional Anal Appl. 2008;13(2):323-352.

[17] Dhage BC. Nonlinear D-set contraction mapping in partially ordered normed linear space and applications to functional hybrid integral equations, Malaya J Mat. 2015;3(1):62-85.

[18] Dhage BC. Lakashmikantham V. On global existence and attractivity results for nonlinear functional integral equations Nonlinear Anal. 2010;72:2219-2227.

[19] Guo D, Lakshmikanatham V, Nonlinear problems in Abstract cones Academic press New York; 1988.

[20] Hashem HHG. Coupled systems of Chandrasekhar's Quadratic integral equations; Electronic Journal Of mathematical Analysis and applications vol1. Jan. 2013;10:1-6.

[21] Heikkila S. Lakshmikantham, Monotone Iterative technique for Discontinuous Nonlinear Differential Equations, Marcel Dekker Inc, New York; 1994.

[22] Dugungi A. Granss, Fixed point theory, Monographic Math Warsaw; 1982.

[23] Banas J, Dhage BC. Globally Asymptotic Stability of solutions of functional integral equations, Nonlinear Functional Analysis.2008;69:1-6.

[24] Karande BD. Fractional order functional integro-differential equation in Banach algebras, Malaysian Journal of Mathematical Science. 2013;8(s):1-16.

[25] Podlubny I, Fractional Differential Equations in mathematics in science and Engineering.Academic press, New York London. 1999;198.

[26] Dhage BC. A nonlinear alternative in Banach algebras with applications to fractional differential equation, Nonlinear Funct. Anal Appl. 2004;8:563-573.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://www.sdiarticle5.com/review-history/95831