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The Importance of the Golden Number for Mathematics and Computer Science: Exploration of the Bergman's System and the Stakhov's Ternary Mirror-symmetrical System (Numeral Systems with Irrational Bases)

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

The main goal of the article is to explore two unusual numeral systems, which alter radically our ideas on the positional numeral systems. We are talking on the numeral systems with *irrational bases*. The first of them is the binary (0,1) numeral system with the irrational base $\Phi=\frac{1+\sqrt{5}}{2}$ (the *golden ratio*), proposed in 1957 by the 12-year American mathematician **George Bergman,** the second is the ternary mirror-symmetrical numeral system with the base $\Phi^2=\frac{3+\sqrt{5}}{2}$, proposed by the author of the present article and published in 2002 in *The Computer Journal* (British Computer Society). *Bergman's system* is the newest mathematical discovery in number theory and the greatest modern mathematical discovery in the field of positional numeral systems after Babylonian numeral system with the base 60, decimal and binary systems. *Bergman's system* can be considered as a new definition of real numbers and is a source of new unusual properties of natural numbers. *Bergman's system* generates the *ternary mirror-symmetrical numeral system*, having unique mathematical property of *mirror symmetry,* which can be used for effective detection of errors in all arithmetical operations. These numeral systems alter our ideas

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about positional numeral systems and can affect on future development of mathematics and computer science. The *ternary mirror-symmetrical numeral system* is possibly the final stage in the long historical development of the concept of *ternary numeral systems*, because in the *ternary mirror-symmetrical numeral system* two scientific problems, the *sign problem and representation of negative numbers* and *problem of error detection*, based on the *principle of mirror symmetry*, are solving simultaneously. The famous American mathematician and expert in computer science **Donald Knut** evaluated highly the *ternary mirror-symmetrical numeral system.* The author is ready to offer consulting services for any electronic company with advanced technology, which can be interested in the technical implementation of the ternary mirror-symmetrical processors and computers on this basis.

Keywords: Decimal system; binary system; ternary system; ternary logic; Brousentsov ternary principle; the golden ratio; Bergman's system; the "golden" number theory; ternary mirror-symmetrical numeral system; ternary mirror-symmetrical summator; matrix ternary mirror-symmetrical summator.

1 Introduction

The present article is development and continuation of author's articles [1,2], published recently in British Journal of Mathematics and Computer Science and author's book **"The Mathematics of Harmony. From Euclid to Contemporary Mathematics and Computer Science** (World Scientific, 2009) [3]. Publication of the articles [1,2] aroused the interest of readers and Editorial Board of BJMCS to the numeral systems with irrational bases. The Mathematics of Harmony [3] as new interdisciplinary direction of modern mathematics and computer science is fruitful source of new ideas in mathematics and computer science, in particular, of numeral systems with irrational bases, which alter our ideas about numeral systems.

As shown in [1], the number of numeral systems with irrational bases, including *Fibonacci p-codes* and *codes of the golden p-proportions*, is theoretically infinite. But a special role in the development of mathematics and computer science could play two new numeral systems with irrational bases: *Bergman's binary system with the base* $\Phi=\frac{1+\sqrt{5}}{2}$ (the *golden ratio*) [4-6] and *Stakhov's ternary mirror-symmetrical system with the base* $\Phi^2=\frac{3+\sqrt{5}}{2}$ [7].

As shown in [7], these numeral systems are closely linked. They may affect on the further development of both mathematics and computer science. Prof. **Donald Knuth**, who is recognized world expert in the field of computer science, has paid special attention to Stakhov's article [7] and promised to include information about this article in the new edition of his best-selling book "The Art of Computer Programming" [8].

The article consists of two parts:

Part I. Bergman's binary numeral system with the base $\Phi=\frac{1+\sqrt{5}}{2}$ and its role for future development of mathematics.

Part II. Stakhov's ternary mirror-symmetrical numeral system with the base $\Phi^2=\frac{3+\sqrt{5}}{2}$ and its role for future development of mathematics and computer science.

Part I. Bergman's binary numeral system with the base $\Phi=\frac{1+\sqrt{5}}{2}$ and its role for future development of mathematics

1 Evolution of Numeral Systems

1.1 The greatest mathematical discovery in mathematics history

Everybody can agree with the statement that any child after graduation from the fourth grade of secondary school should know at least two useful things: the child can read and write on native language and use decimal system to perform elementary arithmetic operations. The *decimal numeral system* is one of the most important achievements of human intellect. This numeral system is based on the "positional principle," introduced by Babylonians in their positional numeral system with the base of 60. The decimal system seems for us such simple and elementary, that it is difficult to agree with the statement that the decimal system is one of the greatest mathematical discoveries in the history of mathematics.

To prove the validity of this statement let us turn to the opinion of the "authorities".

Pierre-Simon Laplace (1749-1827), the famous French mathematician, member of the Parisian academy of sciences, an honourable foreign member of the Petersburg academy of sciences state:

> *"The idea of representation of all numbers by using 9 marks, giving to them, apart from value by the form, another value by the place too, seems so simple what namely because of this simplicity it is difficult to understanding as this is surprising. As not easy to come to this method, we see on the example of the greatest geniuses of Greek science Archimedes and Apollonius, from whom this idea remained latent."*

Fig. 1. Pierre-Simon Laplace
(From Wikipedia, the free encyclopedia https://en.wikipedia.org/wiki/Pierre-Simon_Laplace)

M.V. Ostrogradsky (1801-1862), the famous Russian mathematician, a member of the Petersburg academy of sciences and many foreign academies:

> *"It seems to us that after the invention of written language the largest discovery was the use by humanity of the so-called decimal numeral system. We want to say that the agreement, with the aid of which we can represent all useful numbers by twelve words and by their endings is one of the most remarkable creations of human genius...* "

Fig. 2. M.V. Ostrogradsky
(From Wikipedia, the free encyclopedia https://en.wikipedia.org/wiki/Mikhail_Ostrogradsky)

Jules Tannery (1848-1910), the famous French mathematician, a member of the Parisian academy of sciences:

> *"As to the present system of written numeration in which we use the nine significant numerals and a zero, and the relative value of numerals is defined by a special rule, this system has been introduced in India into the epoch which is not determined precisely, but, apparently, after the Christian era. The invention of this system is one of the most important events in history of science, and despite of a habit to use decimal numeration, we should be surprised by extraordinary simplicity of its mechanism".*

Fig. 3. Jules Tannery
(From Wikipedia, the free encyclopedia https://en.wikipedia.org/wiki/Jules_Tannery)

1.2 A historical view on the positional numeral system

Such a high evaluation of the decimal system in the works of the three famous mathematicians (**Pierre-Simon Laplace, M.V. Ostrogradsky and Jules Tannery**), at first glance seems very strange. But we should not forget that these mathematicians were Great mathematicians-thinkers, who were well aware that the first fundamental results in number theory have been obtained long before the ancient Greeks. These results were obtained for the first time in the theory of *numeral systems*. These include, first of all, the *positional principle of number representation*, embodied in the *Babylonian positional numeral system with the base 60*, and the *doubling principle*, which was used by ancient Egyptians in their *non-positional decimal system* to obtain the rules of multiplication and division of numbers. We should not forget that the *positional principle of number representation* is the basis of all known positional numeral systems, in particular, the *decimal* and *binary*, and that the Egyptian rule of multiplication and division of numbers are used in modern computer arithmetic, based on binary system.

This historical view on the positional numeral system, which goes back to Babylonian and Egyptian mathematics, is used in the present article to assess the role of *Bergman's system* [4-6] and *Stakhov's ternary mirror-symmetrical system* [3,7], in future development of modern mathematics and computer science.

1.3 The main stages in the development of numeral systems

Creation of the first numeral systems is related to the *period of origin of mathematics,* when necessity counting of things, measuring of time, land and quantities of products resulted in development of basic rules of arithmetic for natural numbers. The famous Russian mathematician **Andrey Kolmogorov** (1903 - 1987) emphasizes in his book [9] that *"written numeral systems and common rules of four basic arithmetic operations were gradually developed on the basis of existing oral numeral systems."*

Fig. 4. Academician Andrey Kolmogorov
(*From Wikipedia, the free encyclopedia* https://en.wikipedia.org/wiki/Andrey_Kolmogorov}

In the history of numeral systems we can identify several stages: *initial stage of counting*, stage of *non-positional numeral systems*, *alphabetic numeral systems*, and *positional numeral systems*. Initially people used body parts, fingers, sticks, knots, etc. to represent the counting data. As is outlined in the article [10], "*despite of extreme primitiveness of this method of number representation it had played an exclusive role in the development of concept of number".* This statement confirms that numeral systems played a determining role in formation of concept of *natural numbers*, one of the fundamental concepts of mathematics.

Unfortunately, historians of computer science, being charmed by advanced computer theory, sometimes forget on the role of numeral systems in the history of computers. In fact, the first computing devices ("abacus" and "calculators"), prototypes of modern computers, have been known long before the *Boolean algebra* and *theory of algorithms*. Numeral systems and rules for elementary arithmetical operations played the major role in the development of those primitive computing devices, and we should not forget about this when we try to predict future development of computer technology, especially, computer technology for "mission-critical applications".

Why in number theory there was not paid much attention to numeral systems in comparison to other directions of number theory? Possibly, the mathematical "tradition" had played here negative role. In the

Greek mathematics, which had reached a high level of development, for the first time arose a division of mathematics into *"higher" mathematics*, which had included *geometry* and *theory of numbers*, and *"logistics"*, the applied science for *arithmetic calculations* (including *"school" arithmetic*), *geometric measurements* and *constructions*.

Already since Plato's time, the "logistics" has been considered as applied discipline, which was not included in the educational scope of philosophers and scientists. The scornful attitude to the "school" arithmetic and their problems, starting since Plato, and also an absence of any enough serious need for creation of new numeral systems in computing practice, which had been satisfied by *decimal system* during last centuries, and in the last decades by *binary system* (in computer science), can serve as an explanation of that fact, why theory of numbers showed a small attention to the theory of numeral systems, and why in this part the number theory remained at the level of the Babylonian or Hindi mathematics.

However, an attitude to numeral systems sharply changed in the second half of the 20^{th} century after occurrence of modern computers. In this area has a huge interest to methods of number representation and new computer arithmetic's arose again.

Excepting usage of *binary system* ("von Neumann's principles," developed by famous mathematician an physicist **John von Neumann** (1903-1957)), already at the initial stage of the "computer era", the attempts have been undertaken to use in computers others numeral systems, distinct from binary (in this respect, the ternary computer "Setun" designed by the Russian engineer **Nikolay Brousentsov** (1925-2014) is the most bright example [11].

Fig. 5. John von Neumann
(*From Wikipedia, the free encyclopedia*
https://en.wikipedia.org/wiki/John_von_Neumann

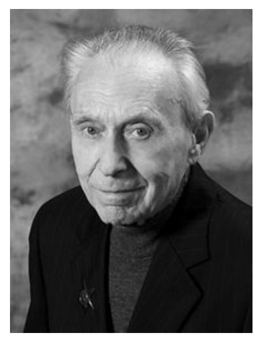

Fig. 6. Nikolay Brousentsov
(*From Wikipedia, the free encyclopedia*
https://ru.wikipedia.org/wiki/%D0%91%D1%80%D1%83%D1%81%D0%B5%D0%BD%D1%86%D0%BE%D0%B2,_%D0%9D%D0%B8%D0%BA%D0%BE%D0%BB%D0%B0%D0%B9_%D0%9F%D0%B5%D1%82%D1%80%D0%BE%D0%B2%D0%B8%D1%87)

During this period the numeral systems with the "exotic" titles and properties have appeared: *system for residual classes*, *ternary symmetrical numeral systems*, *numeral systems with the complex bases, nega-positional, factorial, binomial numeral systems*, etc. [12]. All of them had those or other advantages in comparison with *binary system* and have been directed on improvement of those or other characteristics of computers; some of them became a basis for creation of new computer projects (the ternary computer "Setun" [11], the computer based on system for residual classes and so on).

But there is also other interesting aspect of this problem. Four millennia after the invention by Babylonians of the "positional principle", a period of peculiar "Renaissance" in the field of numeral systems arose.

Due to efforts first of all of experts in the computer field, mathematics as though again had returned back to the period of its origin when the numeral systems had defined a subject and essence of mathematics.

In fact, in opinion of many historians of mathematics, this period was extremely important for development of mathematics. During the *period of origin of mathematics* [9], the main concept of number theory, *natural numbers*, was developed.

But then we can put the following question: possibly the modern numeral systems, created for satisfaction of computer needs, could affect on the development of the concept of number and number theory and in such way could influence not only on the development of computer science, but also of all mathematics.

2 A Role of Bergman's System in Future Development of Mathematics

2.1 Definition and brief history

In 1957 the young American mathematician **George Bergman** published the article *A number system with an irrational base* in the authoritative journal *Mathematics Magazine* [4]. The following sum is called *Bergman's system*:

$$A = \sum_i a_i \Phi^i, \tag{1}$$

where A is any real number, a_i is a binary numeral {0,1} of the i-th digit, i=0,±1,±2,±3,…, , Φ^i is the weight of the i-th digit, and $\Phi = \left(1+\sqrt{5}\right)/2$ is the base of the numeral system (1).

Fig. 7. George Bergman
(From Wikipedia, the free encyclopedia
https://en.wikipedia.org/wiki/George_Bergman*)*

We can get more detailed information about **George Bergman** from Wikipedia article [13]. We can read in [13]: *"**George Mark Bergman** was born on 22 July 1943 in Brooklyn, New York. He … received his PhD from Harvard in 1968, under the direction of John Tate. The year before he had been appointed Assistant Professor of mathematics at the University of California, Berkeley, where he has taught ever since, being promoted to Associate Professor in 1974 and to Professor in 1978. His primary research area is algebra, in particular associative rings, universal algebra, category theory and the construction of counterexamples. Mathematical logic is an additional research area. Bergman officially retired in 2009, but is still teaching."*

It is interesting to note the following. The concept of *Bergman's system* has entered widely into Internet and modern scientific literature. The special article in Wikipedia [5] is dedicated to *Bergman's system.* It is described briefly in WolframMathWorld [6]. **Donald Knuth** refers to Bergman's article [4] in his outstanding book [8]. The special paragraph in author's book [3] is dedicated to *Bergman's system. "The Computer Journal"* (British Computer Society) published in 2002 author's article [7]; this article is based on Bergman's system and is dedicated to the so-called *ternary mirror-symmetrical arithmetic*, which was evaluated highly by Prof. **Donald Knuth**. Thus, the article [7] has been glorified Bergman's name more than other his mathematical works, published in adulthood.

2.2 What is the main distinction between Bergman's system and binary system?

On the face of it, there is not any distinction between the formula (1) for Bergman's system and the formulas for the canonic positional numeral systems, in particular, *binary system*:

$$A=\sum_i a_i 2^i \left(i=0,\pm1,\pm2,\pm3,...\right)\left(a_i\in\{0,1\}\right), \tag{2}$$

where the digit weights are connected by the following "arithmetical" relations:

$$2^i=2^{i-1}+2^{i-1}=2\times2^{i-1}, \tag{3}$$

which underlie "binary arithmetic".

The principal distinction of the numeral system (1) from the binary system (2) is the fact that the irrational number $\Phi=\left(1+\sqrt{5}\right)/2$ (the golden ratio) is used as the base of the numeral system (1) and the digit weights are connected by the following relations:

$$\Phi^i=\Phi^{i-1}+\Phi^{i-2}=\Phi\times\Phi^{i-1}, \tag{4}$$

which underlie the "golden" arithmetic.

That is why; Bergman called his numeral system the *numeral system with irrational base*. Although Bergman's article [7] is a fundamental result for number theory and computer science, mathematicians and engineers of that period were not able to appreciate the mathematical discovery of American wunderkind.

2.3 Unusual mathematical properties of Bergman's system

<u>2.3.1 The "extended" Fibonacci and Lucas numbers</u>

Bergman's system (1) is connected closely with the so-called *"extended" Fibonacci and Lucas numbers* F_i and $L_i\left(i=0,\pm1,\pm2,\pm3,...\right)$. Table 1 gives example of the *"extended" Fibonacci numbers.*

Table 1. The "extended" Fibonacci and Lucas numbers

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	-1	2	-3	5	-8	13	-21	34	-55
L_n	2	1	3	4	7	11	18	29	47	76	123
L_{-n}	2	-1	3	-4	7	-11	18	-29	47	-76	123

From Table 1, there follow the following relations, which connect the "extended" Fibonacci and Lucas numbers:

$$F_{-n}=\left(-1\right)^{n+1}F_n;\quad L_{-n}=\left(-1\right)^n L_n \tag{5}$$

<u>2.3.2 Φ-, F- and L-codes</u>

Let us consider the representation of natural numbers in *Bergman's system*, called Φ*-code of natural number N*:

$$N=\sum_i a_i\Phi^i\left(i=0,\pm1,\pm2,\pm3,...\right). \tag{6}$$

The abridged notation of the Φ-code (6) in the form

$$N = a_n a_{n-1} \ldots a_1 a_0 . a_{-1} a_{-2} \ldots a_{-(k-1)} a_{-k} . \tag{7}$$

is called the *"golden" representation of natural number N.*

The unusual mathematical properties of *Bergman's system* arise at representation of natural numbers in the Φ-code (6). We formulate them in the form of the following theorems [2].

Theorem 1. *All natural numbers can be represented in Φ-code (6) by using a finite number of binary numerals.*

Theorem 2 (Z-property). *If we represent an arbitrary natural number N in the Φ-code (6) and then substitute the "extended" Fibonacci number $F_i\,(i = 0, \pm 1, \pm 2, \pm 3, \ldots)$ instead of the golden ratio powers Φ^i in the expression (6), then the sum that appear as a result of such a substitution is equal to 0 identically, independently on the initial natural number N, that is,*

$$For\ any\ N = \sum_i a_i \Phi^i\ after\ substitution\ F_i \to \Phi^i : \sum_i a_i F_i \equiv 0\,(i = 0, \pm 1, \pm 2, \pm 3, \ldots). \tag{8}$$

Theorem 3 (*D*-property). *If we represent an arbitrary natural number N in the Φ-code (6) and then substitute the Lucas number $L_i\,(i = 0, \pm 1, \pm 2, \pm 3, \ldots)$ instead of the golden ratio powers Φ^i in the formula (6), then the sum that appears as a result of such a substitution is equal to 2N identically, independently of the initial natural number N, that is,*

$$For\ any\ N = \sum_i a_i \Phi^i\ after\ substitution\ F_i \to \Phi^i : \sum_i a_i L_i \equiv 2N\,(i = 0, \pm 1, \pm 2, \pm 3, \ldots). \tag{9}$$

Theorem 4 (*F*-code). *If we represent an arbitrary natural number N in the Φ-code (6) and then substitute the "extended" Fibonacci number $F_{i+1}\,(i = 0, \pm 1, \pm 2, \pm 3, \ldots)$ instead of the golden ratio powers Φ^i in the expression (6), then the sum that appear as a result of such a substitution is equal to N identically, independently on the initial natural number N, that is,*

$$N = \sum_i a_i F_{i+1}\,(i = 0, \pm 1, \pm 2, \pm 3, \ldots). \tag{10}$$

The expression (10) is named the *F-code of natural number N* [2].

Theorem 5 (*L*-code). *If we represent an arbitrary natural number N in the Φ-code (6) and then substitute the "extended" Lucas number $L_{i+1}\,(i = 0, \pm 1, \pm 2, \pm 3, \ldots)$ instead of the golden ratio powers Φ^i in the expression (6), then the sum that appear as a result of such a substitution is equal to N identically, independently on the initial natural number N, that is,*

$$N = \sum_i a_i L_{i+1}\,(i = 0, \pm 1, \pm 2, \pm 3, \ldots). \tag{11}$$

The expression (11) is named the *L-code of natural number N* [2].

As the binary numerals $a_i \in \{0,1\}$ in the expressions (6), (10) and (11) coincide, this means that there are three different methods of positional representation of one and the same natural number *N*: Φ*-code* (6), *F-code* (10) and *L*-code (11).

Let us consider the representation of the sums (6), (10) and (11) in the abridged form (7) called the "golden" representation of natural number *N*. It is clear that the Φ-code (6), *F*-code (10) and *L*-code (11) lead us to one and the same "golden" representation of natural number *N* in the form (7) because all the binary numerals $a_i \in \{0,1\}$ in the expressions (6), (10) and (11) coincide.

2.3.3 A numerical example

Once again let us consider the abridged representation (7). We can see that the "golden" representation (7) is divided by the comma into two parts, namely the left-hand part, which consists of the digits with non-negative indices, and the right-hand part, which consists of the digits with negative indices. For example, we can consider the "golden" representation of the decimal number 10 in Bergman's system:

$$10_{10} = 10100.0101_{\Phi} \tag{12}$$

For the Φ-code (6) the "golden" representation (12) has the following numerical interpretation:

$$10_{10} = \Phi^4 + \Phi^2 + \Phi^{-2} + \Phi^{-4} \tag{13}$$

By using the well-known formula $\Phi^i = \frac{L_i + F_i\sqrt{5}}{2}$, we can represent the sum (13) as follows:

$$10_{10} = \frac{L_4 + F_4\sqrt{5}}{2} + \frac{L_2 + F_2\sqrt{5}}{2} + \frac{L_{-2} + F_{-2}\sqrt{5}}{2} + \frac{L_{-4} + F_{-4}\sqrt{5}}{2}. \tag{14}$$

If we take into consideration the relations (5), which connect the "extended" Fibonacci and Lucas numbers we can reduce the expression (14) to the following:

$$10_{10} = \frac{2(L_4 + L_2)}{2} = L_4 + L_2 = 7 + 3.$$

Now, let us consider the interpretation of the "golden" representation (7) as the *F*- and *L*-codes:

$10_{10}=F_5+F_3+F_{-1}+F_{-3}=5+2+1+2$ (***F-code***);
$10_{10}=L_5+L_3+L_{-1}+L_{-3}=11+4-1-4$ (***L-code***) .

Also we can check the sum (14) according to the *Z*- and *D*-properties. If we substitute in (14) the "extended" Fibonacci and Lucas numbers F_i and L_i instead the powers Φ^i, we obtain the following sums:

$F_4+F_2+F_{-2}+F_{-4}=3+1+(-1)+(-4)=0$ (***Z-property***);
$L_4+L_2+L_{-2}+L_{-4}=7+3+3+7=20=2\times 10$ (***D-property***).

2.3.4 Shifting Φ-, F- and L-codes

Once again, we note that the *Φ*-, *F*- and *L*-codes (6), (10), (11) have one and the same "golden" representation (7) of the given natural number *N*. However, a difference between them appears when we start to shift the "golden" representation (7) to the right or to the left.

Let us denote by $N_{(k)}$ and $N_{(-k)}$ the results of shifting the "golden" representation (7) on the *k* digits to the left and to the right, respectively.

If we interpret the "golden" representation (7) as the *Φ-code of natural number N* given by (6), then its shifting to the left (that is, to the side of the highest digits) by one digit corresponds to the multiplication of the number *N* by the base Φ (the golden ratio), and its shifting to the right (that is, to the side of the lowest digits) by one digit corresponds to the division of the number *N* by the base Φ (the golden ratio), that is,

$$N_{(1)} = N \times \Phi = \sum_i a_i \Phi^{i+1} \tag{15}$$

$$N_{(-1)} = N \times \Phi^{-1} = \sum_i a_i \Phi^{i-1} . \tag{16}$$

It is clear that shifting the "golden" representation (7) on the *k* digits to the left corresponds to the multiplication of the number *N* by Φ^k and its shifting on the *k* digits to the right corresponds to the division by Φ^k , that is,

$$N_{(k)} = N \times \Phi^k = \sum_i a_i \Phi^{i+k} \tag{17}$$

$$N_{(-k)} = N \times \Phi^{-k} = \sum_i a_i \Phi^{i-k} . \tag{18}$$

Consider shifting the "golden" representation (7) to the left and to the right when we interpret it as the *F*- or *L*-codes. Results of these code transformations are given by the following theorems proved in [3].

Theorem 6. *Shifting the "golden" representation (7), which is interpreted as the F-code (10), on the k digits to the left (that is, to the side of the highest digits) corresponds to the multiplication of the number N by the "extended" Fibonacci number* F_{k+1}*; however, its shifting on the k digits to the right (that is, to the side of the lowest digits) corresponds to the multiplication of the number N by the "extended" Fibonacci number* F_{-k+1}*, that is,*

$$N_{(k)} = \sum_i a_i F_{i+k+1} = F_{k+1} \times N \tag{19}$$

$$N_{(-k)} = \sum_i a_i F_{i-k+1} = F_{-k+1} \times N. \tag{20}$$

Theorem 7. *Shifting the "golden" representation (7), which is interpreted as the L-code (11), on the k digits to the left (that is, to the side of the highest digits) corresponds to the multiplication of the number N by the "extended" Lucas number* L_{k+1}*; however, its shifting on the k digits to the right (that is, to the side of the lowest digits) corresponds to the multiplication of the number N by the "extended" Lucas number* L_{-k+1}*, that is,*

$$N_{(k)} = \sum_i a_i L_{i+k+1} = L_{k+1} \times N \tag{21}$$

$$N_{(-k)} = \sum_i a_i L_{i-k+1} = L_{-k+1} \times N. \quad (22)$$

Note that the proof of Theorems 6, 7 was made in [3] by using the following well-known identities:

$$F_{i+1+k}=F_k F_i+F_{k+1}F_{i+1} \quad (23)$$

$$L_{i+k+1}=L_k F_i+ L_{k+1}F_{i+1} \quad (24)$$

In conclusion, we note that Theorems 1-7 are true only for natural numbers. This means that Theorems 1-7 express new fundamental properties of natural numbers. It is surprising for many mathematicians to know that the new mathematical properties of natural numbers were only discovered at the beginning of the 21th century, that is, 2½ millennia after the beginning of their theoretical study. The golden ratio and the "extended" Fibonacci and Lucas numbers play a fundamental role in this discovery. This discovery connects together two outstanding mathematical concepts of Greek mathematics - *natural numbers* and *golden ratio.* This discovery is the confirmation of fundamental role of *Bergman system* (1) in development of the "golden" number theory, described in [2].

2.4 Pythagorean MATHEM's and historical role of Bergman's system for future development of mathematics and computer science

The famous American mathematician and thinker Prof. **Donald Knuth**, who is the best world expert in computer science, became possibly the first famous scientist in computer science, who evaluated *Bergman's system*. He referred to Bergman's 1957 article [7] in Volume 1 of his world bestseller *Art of Computer Programming* [8] and this fact by itself is high evaluation of the important role of *Bergman's system* for the development of computer science.

But *Bergman's system* is also of great importance for the future development of mathematics. To substantiate this claim, we turn to the history of Greek mathematics, starting from **Pythagoras, Plato** and **Euclid**.

2.4.1 Pythagoreanism and Pythagorean MATHEM's

We begin from *Pythagorean MATHEM's.* By studying the sources of mathematics, we inevitably come to **Pythagoras** and his doctrine, called *Pythagoreanism* [14]. As highlighted in [14], "*Pythagoreanism was originated in the 6th century BCE, based on the teachings and beliefs held by Pythagoras and his followers, the Pythagoreans, who were considerably influenced by mathematics and mysticism. Later revivals of Pythagorean doctrines led to what is now called Neopythagoreanism or Neoplatonism. Pythagorean ideas exercised a marked influence on Aristotle, and Plato, and through them, all of Western philosophy*".

According to tradition, *Pythagoreans* were divided into two separate schools of thought, the *mathēmatikoi* (*mathematicians*) and the *akousmatikoi* (*listeners*). *Listeners* developed religious and ritual aspects of *Pythagoreanism*, *mathematicians* studied four Pythagorean MATHEM's: *arithmetic, geometry*, *harmonics* and *spherics*. **These MATHEM's, according to Pythagoras, were the main component parts of Mathematics.**

2.4.2 Platonic Solids, golden ratio and Plato's cosmology

Pythagorean philosophy had a huge impact on *Plato's studies*. Of particular interest was the application of the *Platonic Solids* and *golden ratio*, borrowed by **Plato** from the Pythagorean geometry and number theory, in *Plato's cosmology*.

The famous Russian thinker **Alexey Losev** (1893-1988), philosopher of the aesthetics of antiquity and the Renaissance, evaluates the main achievements of the ancient Greeks in this area as follows [15]:

> *"From Plato's point of view, and in general in terms of the entire ancient cosmology, the Universe is determined as a certain proportional whole that obeys the law of harmonic division - the golden section .... The ancient Greek system of cosmic proportions in the literature is often interpreted as a curious result of unrestrained and wild imagination. In such explanations we can see the unscientific helplessness of those who claim it. However, we can understand this historical and aesthetic phenomenon only in connection with a holistic understanding of history, that is, by using a dialectical view of culture and looking for the answer in the peculiarities of ancient social life."*

Here **Losev** formulates the *"golden" paradigm of ancient cosmology,* based on the *golden ratio*. It is based upon the most important ideas of ancient science that are sometimes treated in modern science as a *"curious result of an unrestrained and wild imagination."* First of all, we are talking about the *Pythagorean Doctrine of the Numerical Harmony of the Universe* and *Plato's cosmology based on the Platonic solids*. By referring to the geometrical structure of the Cosmos and its arithmetical relations, expressing *Cosmic Harmony*, the Pythagoreans anticipated the modern mathematical basis of the natural sciences, which began to develop rapidly in the 20th century. Pythagoras's and Plato's ideas about Cosmic Harmony has proven to be immortal.

Johannes Kepler (1571-1630), the famous astronomer and author of *Kepler's laws*, expressed his admiration for the golden ratio in the following words (cited from [16]):

> *"Geometry has two great treasures: one is the theorem of Pythagoras; the other the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel."*

You will recall that the ancient problem of the *division in extreme and mean ratio* (DEMR) was Euclid's language for the *golden ratio*!

Fig. 8. Johannes Kepler
(*From Wikipedia, the free encyclopedia https://en.wikipedia.org/wiki/Johannes_Kepler*)

Fig. 9. Alexey Losev
(*From Wikipedia, the free encyclopedia https://en.wikipedia.org/wiki/Aleksei_Losev*)

2.4.3 Euclid's Elements and Proclus hypothesis

As it is well-known, Euclid's *Elements* had an enormous influence on the development of not only geometry, but on the entire science in general. As it is emphasized in Wikipedia [17], "*the Elements are still considered a masterpiece in the application of logic to mathematics. In historical context, it has proven enormously influential in many areas of science. Scientists Nicolaus Copernicus, Johannes Kepler, Galileo Galilei, and Sir Isaac Newton were all influenced by the Elements, and applied their knowledge of it to their work. Mathematicians and philosophers, such as Bertrand Russell, Alfred North Whitehead, and Baruch Spinoza, have attempted to create their own foundational "Elements" for their respective disciplines, by adopting the axiomatized deductive structures that Euclid's work introduced.*"

Discussing Euclid's *Elements*, we always ask the question: for what purpose Euclid wrote his *Elements*? At first glance, it seems that the answer to this question is very simple: The main purpose of Euclid was to set forth the main achievements of the Greek mathematics for 300 years prior to Euclid, by using the "axiomatic method." Indeed, Euclid's *Elements* is the main work of ancient Greek science, devoted to the axiomatic description of geometry and mathematics, based on "axiomatic method." This view on the *Elements* is most common in mathematics.

But, besides the "axiomatic" point of view, there is another point of view on the motives of Euclid at writing the *Elements*. For the first time, this point of view has been substantiated by **Proclus Diadochus** (412-485) [18], the Neoplatonic philosopher and mathematician, one of the first commentators of Euclid's *Elements*.

Detailed analysis of *Proclus hypothesis* is given in author's article [19]. The following unexpected conclusion follows from *Proclus hypothesis*. Proclus says that the main goal of *Euclid* to write his *Elements* was creating complete theory of the five *Platonic solids,* which expressed the *Universe Harmony* in *Plato's cosmology*. Euclid posted this theory in the XIII-th, that is, final Book of the *Elements*. For the construction of the geometric theory of the *dodecahedron*, whose faces are *regular pentagons*, Euclid described the golden ratio in the Book II (Proposition II.11). As it follows from *Proclus hypothesis*, in parallel with *Classical Mathematics*, another mathematical discipline, the *Mathematics of Harmony*, begun to develop in mathematics since Euclid's *Elements*, which are the source for both directions.

However, the *Classical Mathematics* borrowed the axiomatic approach and other ancient mathematical achievements (number theory, theory of irrationalities and so on), while the *Mathematics of Harmony* borrowed the *golden ratio* (Proposition II.11) and *Platonic Solids*, described in Book XIII of Euclid's *Elements*.

By returning again to the Pythagorean mathematics, it should be noted that all parts of mathematics (*arithmetics, geometry*, *harmonics* and *spherics*) presented in Euclid's *Elements*.

It should be noted two important conclusions arising from Proclus hypothesis:

1. Unfortunately, in the process of historical development, one important part of mathematics – *harmonics,* which in the interpretation of **Pythagoras, Plato** and **Euclid** was connected with *Platonic solids* and *golden ratio*, disappeared from mathematics. Possibly, author's book "Mathematics of Harmony" [3] is the first attempt in modern mathematics to regenerate this important Pythagorean MATEM in mathematics.
2. The idea of the Universe Harmony on the basis of the *Platonic solids* and *golden ratio*, begins to revive actively in modern theoretical natural sciences. This is confirmed by the latest outstanding discoveries in chemistry, crystallography, botany and other sciences, in particular, by *fullerenes* (Nobel Prize - 1996) [20] and *quasi-crystals* (Nobel Prize - 2011) [21], whose symmetry is related to Platonic solids, and also by *new geometrical theory of phyllotaxis* ("Bodnar's geometry") [22, 23], based on the *golden ratio* and the "golden" recursive hyperbolic functions [24]. These examples show that the great predictions by **Pythagoras, Plato** and **Euclid** are true for modern theoretical natural sciences. However, many modern mathematicians continue to consider "harmonic ideas" by **Pythagoras, Plato** and **Euclid** as "*curious result of unrestrained and wild imagination"* (Alexey Losev).

Modern historians of mathematics refer to Proclus hypothesis sufficiently cautiously and sometimes simply ignore it [9]. This is not consistent with the opinion of other historians of mathematics [25-28]. For example, in comments to Euclid's *Elements* [28], Prof. **D.D. Mordukhai-Boltovskii** (1876-192), the authoritative Russian historian of mathematics and translator of Euclid's *Elements* into Russian [28]) writes the following:

> *"After the careful analysis of Euclid's Elements, I have been convinced firmly that the construction of regular polyhedra, and even more - the proof of the existence of five and only five regular polyhedra - represented the ultimate goal of the work, which led to the origin of the Elements."*

If we accept this hypothesis, then our views on Euclid's *Elements* and the entire history of mathematics, starting since Euclid, completely can be changed what can affect on the structure of mathematics, including mathematical education.

In addition, it follows from *Proclus hypothesis* that Euclid's *Elements* are the first in mathematics history **Mathematical theory of the Universe Harmony**, based on *Platonic solids*. The modern *Mathematics of Harmony* [3] should be recognized as an important part of mathematics, which revives the Pythagorean MATEM of *harmonics*, lost in the process of historical development of mathematics.

Part II. Stakhov's Ternary Mirror-symmetrical Numeral System with the Base $\Phi^2 = \frac{3+\sqrt{5}}{2}$ and Its Role for Future Development of Mathematics and Computer Science

Despite the fact that the ternary mirror-symmetrical numeral system has been described by the author in the article [7], the author considers it appropriate to give brief description of this unusual positional numeral system. In the present article, the author emphasizes fundamental relationship of the ternary mirror-symmetrical numeral system with Bergman's system and considers it as an important part of the "golden" number theory [2]. Also the author draws attention to new technical solutions in this field, which may be of interest for many computer experts.

1 Ternary Numeral Systems and Ternary Principle of Nikolay Brousentsov

1.1 Preliminary information

As is well known, computer process design begins with the choice of numeral system that determines many technical characteristics of computers. At the beginning of the computer era, the problem of choosing the "optimal" number system for electronic computers was brilliantly solved by the American physicist and mathematician **John von Neumann**, who forcefully argued his preference for the *binary system* in electronic computers. The famous *John von Neumann Principles* include three basic ideas for electronic computer design: the *Binary System, Binary (Boolean) Logic,* and *the Binary Memory Element ("Flip-Flop).*

Even though the binary system is the most popular in contemporary computers, the studies and developments of new numeral systems is continuing. The desire to overcome a number of significant shortcomings of the classical binary system is the primary motivation for these ongoing studies. Two shortcomings of the binary system are certainly well known. The first of them involves the fact that it is impossible to represent negative numbers (the *sign problem*) and perform arithmetical operations on them in "direct" binary code what complicates arithmetical computer structures.

The *problem of "zero" redundancy* is the second shortcoming of the binary system. The fact, that all binary combinations are allowed, is the reason why errors can not be detected at information transmission, processing, and storage.

The initial attempt to overcome the *sign problem* was made in the Soviet Union during the very dawn of the computer era. The original computer project – the ternary computer "Setun" [11] – was designed in 1958 at Moscow University, and became a brilliant example for the "optimal" solution of the *sign problem.* A new principle of construction for computers was implemented in the ternary computer "Setun." This principle was based on the concepts of *ternary logic, ternary symmetrical numeral system,* and *ternary memory element ("flip-flap-flop").* This principle is called *Brousentsov's ternary principle* [7] in honor of the Soviet scientist **Nikolay Brousentsov** (1925-2014), the principal designer of the "Setun" computer.

1.2 Ternary-symmetrical numeral system and ternary logic

1.2.1 Ternary-symmetrical numeral system

The *ternary symmetrical numeral system* is the key idea of *Brousentsov's ternary principle*:

$$N = \sum_{i=1}^{n} b_i 3^{i-1} , \tag{25}$$

where b_i (i = 1, 2, …, n) is the ternary numeral $\{\bar{1},0,1\}$ of the i-th digit; 3^{i-1} is the "weight" of the i-th digit; the number 3 is the base of the numeral system.

The basic advantage of the numeral system (25) in the comparison to the classical binary system with the numerals 0 and 1 is a graceful solution of the "*sign problem*". A sign of the number is determined by the highest significant numeral of the ternary symmetric representation (25). For example, the number $N_1 = 0\ \bar{1}\ 1\ 1\ 0\ \bar{1}$ is negative but the number $N_2 = 1\ \bar{1}\ 0\ \bar{1}\ 0\ 0\ 1$ is positive. Both positive and negative numbers are represented in the "direct" code and all arithmetical operations are fulfilled in the "direct" code. It is easy to get the representation of the negative number (-N) from the ternary representation of the positive number N using the rule of ternary *inversion*:

$$1 \to \bar{1},\quad 0 \to 0,\quad \bar{1} \to 1. \tag{26}$$

1.2.2 The basic functions of the ternary logic

The ternary logic is a special case of the so-called k-valued logic (k=2, 3, 4, 5, …) for the case k=3. For the coordination with the ternary symmetrical numeral system we will assume that the ternary logical variables take their values from the set $\{\bar{1}, 0, 1\}$.

Then the basic logical functions of *one ternary variable* ν are determined in the following manner:

Inversion function	**Cyclic negation**
$f(v) = \bar{v} = \begin{cases} \bar{1}\ with\ \ v = 1 \\ 0\ with\ \ v = 0 \\ 1\ with\ \ v = \bar{1} \end{cases}$	$f(v) = \tilde{\tilde{v}} = \begin{cases} \bar{1}\ with\ \ v = 0 \\ 0\ with\ \ v = 1 \\ 1\ with\ \ v = \bar{1} \end{cases}$

Consider the following important functions of *two ternary variables*:

(1) *Ternary conjunction* $f(v_1,\ v_2) = min(v_1, v_2) = v_1 \wedge v_2$

$\wedge$	$\bar{1}$	0	1
$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$
0	$\bar{1}$	0	0
1	$\bar{1}$	0	1

(2) *Ternary disjunction* $f(v_1, v_2) = max(v_1, v_2) = v_1 \vee v_2$

$\vee$	$\bar{1}$	0	1
$\bar{1}$	$\bar{1}$	0	1
0	0	0	1
1	1	1	1

(3) *Addition by modulo 3* $f(v_1, v_2) = v_1 \oplus v_2 \ (mod \ 3)$

$\oplus$	$\bar{1}$	0	1
$\bar{1}$	1	$\bar{1}$	0
0	$\bar{1}$	0	1
1	0	1	$\bar{1}$

(4) *Multiplication by modulo 3* $f(v_1, v_2) = v_1 \otimes v_2 \ (mod \ 3)$

$\otimes$	$\bar{1}$	0	1
$\bar{1}$	1	0	$\bar{1}$
0	0	0	0
1	$\bar{1}$	0	1

1.2.3 The binary realization of the ternary logical elements

For the micro-electronic realization by using VLSI we can use the following binary coding of the ternary variables as is shown in Table 2.

Table 2. Binary coding (x_1, x_2) of the ternary numerals *v*

v	x_1	x_2
$\bar{1}$	1	0
0	0	0
1	0	1

Using Table 2 each ternary element can be represented by means of VLSI with the binary inputs and outputs. Then the problem of designing ternary elements is reduced to designing binary VLSI.

Note that some ternary functions are realized very simply in this manner. For example the logical element of the “ternary inversion” $f(v) = \bar{v}$ ($v = x_1x_2$ and $\bar{v} = x_2x_1$) is realized as shown in Fig. 10.

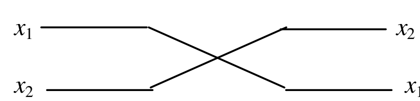

Fig. 10. The binary realization of the ternary inversion

Flip-flap-flop

The same "binary approach" can be used for designing the ternary memory element called *flip-flap-flop*. As is well known, the classical binary *flip-flop* is based on the logical elements 1 and 2 of the kind *OR-NOT* (Fig. 11-a), which are connected by the back logical connections.

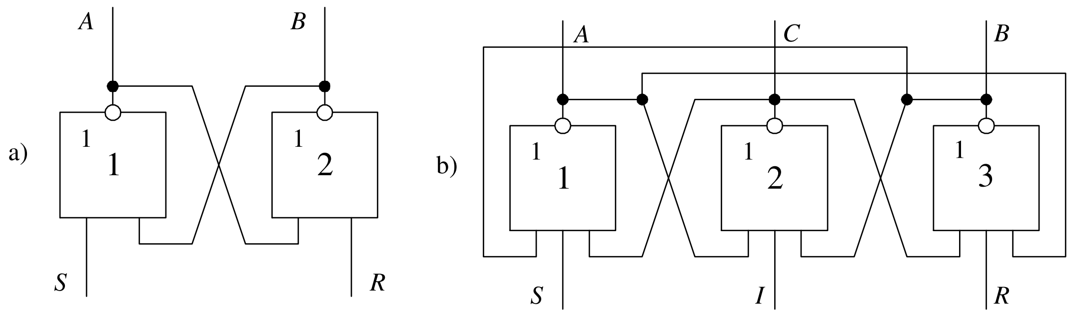

Fig. 11. "Flip-flop" (a) and "flip-flap-flop" (b)

Consider now the logical circuit, which consists of the three logical elements 1, 2, 3 of the kind *OR-NOT* (Fig. 11-b). Suppose that the logical elements 2 and 3 are adjacent to the logical element 1, the logical elements 3 and 1 are adjacent to the logical element 2, and the logical elements 1 and 2 are adjacent to the logical element 3. Each logical element *OR-NOT* is connected with its adjacent logical elements by the back logical connections. This is a cause of the three stable states of the logical circuit in Fig. 11-b. In fact, suppose that we have the logical 1 on the input *C* of the logical element 2. This logical 1 enters the inputs of the adjacent logical elements 2 and 3 and supports the logical 0 on their outputs *A* and *B*. These logical 0's enter the inputs of the logical element 2 and support the logical 1 on its output *C*. Hence, this state of the circuit in Fig. 11-b is the first stable state. This stable state corresponds to the code combination 0 1 0 on the outputs *A, C, B*. One may show that the circuit has one more two stable states corresponding to the code combinations 1 0 0 and 0 0 1 on the outputs *A, C, B*. In fact, it is easy to show that the logical 1 on the output *A* is a cause of the second stable state 100 of the logical circuit in Fig. 11-b. At least, the logical 1 on the output *B* is a cause of the third stable state of the logical circuit in Fig. 11-b. We can use the above-mentioned stable states of the circuit in Fig. 11-b for the binary coding of the ternary numerals according to the following table:

$$0 = 0\ 1\ 0$$
$$1 = 0\ 0\ 1$$
$$\bar{1} = 1\ 0\ 0$$

If we eliminate the middle output *C* we will get the binary outputs *A* and *B*, which correspond to the binary coding of the ternary variables according to Table 2.

Hence, the logical circuit in Fig. 11-b can be considered as the ternary-binary memory element called *flip-flap-flop*. Consider now the functioning the "flip-flap-flop" in Fig. 11-b. It has three stable states $\bar{1}$, 0 and 1. Let the "flip-flap-flop" in Fig. 11-b be in the state $Q = 0$. This means that the output $C = 1$, and other outputs $A = B = 0$. If we need to set the "flip-flap-flop" into the state $Q = 1$ (0 0 1) we have to send to the "flip-flap-flop" inputs *S*, *I, R* the following adjusting signals $S = 1$, $I = 1$, $R = 0$. The signals $S = 1$ and $I = 1$ cause the appearance of the logical 0's on the outputs *A* and *C*. These logical 0's enter the inputs of the logical element 3 and together with the logical signal $R = 0$ cause an appearance of the logical 1 on the output *B*.

By analogy one may show that the adjusting signals $S = 0$, $I = 1$, $R = 1$ turn the "flip-flap-flop" in Fig. 11-b into the state $\bar{1}$ (100).

2 Ternary Mirror-symmetrical Numeral System

2.1 Conversion of the binary "golden" representation to the ternary "golden" representation

We start from the binary Φ-code (6). We will use the MINIMAL FORM of the Φ-code (6). This means that each binary unit $a_k = 1$ in the binary "golden" representation (7) would be "enclosed" by the two next binary zeros $a_{k-1} = a_{k+1} = 0$.

Consider now the following identity for the powers of the golden ratio:

$$\Phi^k = \Phi^{k+1} - \Phi^{k-1}. \tag{27}$$

The identity (24) has the following code interpretation:

$$\begin{matrix} k+1 & k & k-1 & & k-1 & k & k-1 \\ 0 & 1 & 0 & = & 1 & 0 & \bar{1} \end{matrix} \tag{28}$$

where $\bar{1}$ is the negative unit, that is, $\bar{1} = -1$. It follows from (25) that the positive binary 1 of the k^{th} digit is transformed into two 1's, the positive unit 1 of the $(k+1)^{\text{th}}$ digit and the negative unit $\bar{1}$ of the $(k-1)^{\text{th}}$ digit.

The code transformation (28) can be used for the conversion of the MINIMAL FORM of the binary "golden" representation (7) of the number N into the *ternary "golden" representation* of the same integer N.

As a result of such conversion we get the following sum:

$$N = \sum_i c_i \Phi^{2i}, \tag{29}$$

where c_i is the ternary numeral of the i^{th} digit; Φ^{2i} is the weight of the i^{th} digit; Φ^2 is the base of the numeral system (29).

The sum (26) is called *ternary Φ-code of natural number N*. The abridged notation of the *ternary Φ-code of natural number N* has the following form:

$$N = c_k c_{k-1} \ldots c_2 c_1 c_0 c_{-1} c_{-2} \ldots c_{-(k-1)} c_{-k}, \tag{30}$$

where $c_i \in \{\bar{1}, 0, 1\} (i = 0, \pm1, \pm2, \pm3, \ldots)$ is ternary numeral (*treat*) of i-th digit.

The notation (30) is called the *ternary "golden" representation of natural number N.*

Note that the *ternary "golden" representation* (30) consists of two parts separated by 0-th digit c_0: The left part $c_k c_{k-1} \ldots c_2 c_1$ and the right part $c_{-1} c_{-2} \ldots c_{-(k-1)} c_{-k}$.

2.2 The ternary *F*- and *L*-codes

2.2.1 Definition

Above we have introduced the so-called binary *F*- and *L*-codes (10) and (11). Recall that these unusual codes are equivalent of the Φ-code (6) of the same natural number *N*. Using the *ternary Φ-code of natural number N* (29), by analogy we can introduce the ternary *F*- and *L*-codes of the same natural number *N* in the following forms:

$$N = \sum_i c_i F_{2i+1} \tag{31}$$

$$N = \sum_i c_i L_{2i+1} \,. \tag{32}$$

Note that the values of the ternary digits in the codes (29), (31), (32) coincide.

2.2.2 Property of "mirror symmetry"

Table 3 demonstrates examples of *the "golden" ternary representations* of integers from 0 to 10.

By studying Table 3, we find unexpected fundamental property for all the *"golden" ternary representations*, represented in Table 3: The left part of the *golden" ternary representation* of the given integer *N* relatively to 0th digit is *mirror-symmetrical* to its right part. This property is called the property of *mirror symmetry.*

Table 3. "Golden" ternary representations

i	**3**	**2**	**1**	**0**	**-1**	**-2**	**-3**
Φ^{2i}	Φ^{6}	Φ^{4}	Φ^{2}	Φ^{0}	Φ^{-2}	Φ^{-4}	Φ^{-6}
F_{2i+1}	13	5	2	1	1	2	5
L_{2i+1}	29	11	4	1	-1	-4	-11
N							
0	0	0	0	0,	0	0	0
1	0	0	0	1,	0	0	0
2	0	0	1	$\bar{1}$,	1	0	0
3	0	0	1	0,	1	0	0
4	0	0	1	1,	1	0	0
5	0	1	$\bar{1}$	1,	$\bar{1}$	1	0
6	0	1	0	$\bar{1}$,	0	1	0
7	0	1	0	0,	0	1	0
8	0	1	0	1,	0	1	0
9	0	1	1	$\bar{1}$,	1	1	0
10	0	1	1	0,	1	1	0

For the case (30), the property of *mirror symmetry* looks as follows:

$$c_k = c_{-k}; c_{k-1} = c_{-(k-1)}; \ldots; c_2 = c_{-2}; c_1 = c_{-1} \,. \tag{33}$$

The violation of at least one equality in (33) is an indication of errors in the *ternary "golden" representation of natural number N.*

Thus, thanks to this simple consideration, we have discovered one more fundamental property of integers called *"mirror-symmetrical property of natural numbers*. Basing on this fundamental property, the "ternary numeral system" given by (29) is called a *ternary mirror-symmetric numeral system* [7].

2.2.3 The base of the ternary mirror-symmetric numeral system

It follows from (29) that the base of this numeral system (26) is the square of the *golden ratio*, that is,

$$\Phi^2 = \frac{3+\sqrt{5}}{2} \approx 2.618\,. \tag{34}$$

This means that the numeral system (29) is a numeral system with irrational base.

The base of the numeral system (29) has the following traditional representation:

$\Phi^2 = 10.$

2.2.4 Representation of negative numbers

The ternary mirror-symmetrical numeral system (29) is similar to the classical ternary-symmetrical numeral system (25) and save the most important advantage of the numeral system (25), which consists in possibility representing both positive and negative numbers in the "direct" code and to perform all arithmetical operations in the "direct" code. The "golden" ternary representation of the negative number ($-N$) can be obtained from the "golden" ternary representation (30) of the initial positive integer N by using of the rule of the *ternary inversion* (26).

2.3 Ternary mirror-symmetrical arithmetic

2.3.1 Comparison of the ternary mirror-symmetric numbers

Consider the set of the weights of the ($2n$ -1)-th ternary mirror-symmetrical representation (30):

$\{\Phi^{2n}, \Phi^{2(n-1)}, \ldots, \Phi^2, \Phi^0, \Phi^{-2}, \ldots, \Phi^{-2(n-1)}, \Phi^{-2n}\}.$

It is easy to prove that the weight of the n-th digit of the "golden" ternary representation (30) is always strictly more that the sum of the rest weights of the representation (30). It follows from this fact that the higher significant digit of the representation (30) contains in itself the information about the sign of the ternary mirror-symmetric number. If the numeral of the higher significant digit of the ternary mirror-symmetrical representation is equal to 1, this means that the ternary mirror-symmetric number is positive. If the numeral of the higher significant digit of the ternary mirror-symmetric representation is equal to $\bar{1}$, this means that the ternary mirror-symmetric number is negative.

It follows from this consideration very simple method of comparison of the two ternary mirror-symmetric numbers A and B by value. The comparison begins from the higher digits of the comparable numbers and lasts before obtaining the first pair of the not coincident ternary digits a_k and b_k. If the numeral $a_k > b_k$ ($1>0$, $1>\bar{1}$, $0>\bar{1}$), then $A>B$. In the opposite case: $A<B$.

Hence, we have found two important advantages of the ternary mirror-symmetric representation (30):

1. Similarly to the classical ternary-symmetric representation (25) the sign of the ternary mirror-symmetrical number is determined by the higher significant digit of the representation (30).

2. Comparison of the numbers is performed similarly to the classical ternary symmetric representation (25), that is, starting from the higher digits before obtaining the first pair of the not coincident ternary digits.

2.3.2 The range of number representation of the ternary mirror-symmetrical numeral system

Consider now the range of number representation in the numeral system (29). Suppose that the ternary "golden" representation (30) has $2m$+1 ternary digits. In this case, by using (30) we can represent all integers in the range from

$$N_{\max} = \underbrace{11...1}_{m}1.\underbrace{11...1}_{m} \tag{35}$$

to

$$N_{\min} = \underbrace{\bar{1}\bar{1}...\bar{1}}_{m}\bar{1}.\underbrace{\bar{1}\bar{1}...\bar{1}}_{m}\ . \tag{36}$$

It is clear that N_{min} is the ternary inversion of N_{max}, that is,

$$|\ N_{min}\ | = N_{max}\ .$$

It follows from this consideration that using the ($2m$+1) ternary digits we can represent in the numeral system (29).

$$2\ N_{max} + 1 \tag{37}$$

integers from N_{max} to N_{min} including the number 0.

For the calculation of N_{max} we can interpret (35) as the ternary L-code (32). Then we can interpret the abridged notation (35) as the following sum:

$$N_{max} = L_{2m+1} + L_{2m\ -1} + ... + L_3 + L_1 + L_{-1} + L_{-3} + ... + L_{-2m+1}\ . \tag{38}$$

For the odd indices i=2k - 1 we have the following property for Lucas numbers [3]:

$$L_{-2m+1} = -\ L_{2m\ -1}\ . \tag{39}$$

Taking into consideration the property (39) we can get the following value of the sum (38):

$$N_{max} = L_{2m+1}. \tag{40}$$

Taking into consideration (37) and (40), we can formulate the following theorem.

Theorem 8. *By using (2m+1) ternary digits in the ternary mirror-symmetric numeral system (29), we can represent* $2L_{2m+1} + 1$ *integers in the range from* $-L_{2m+1}$ *to* L_{2m+1}, *where* L_{2m+1} *is Lucas number*.

2.3.3 Ternary mirror-symmetrical summation

The following identities for the golden ratio powers underlie the mirror-symmetric summation:

$$2\Phi^{2k} = \Phi^{2(k+1)} - \Phi^{2k} + \Phi^{2(k-1)} \tag{41}$$

$$3\Phi^{2k} = \Phi^{2(k+1)} + 0 + \Phi^{2(k-1)} \tag{42}$$

$$4\Phi^{2k} = \Phi^{2(k+1)} + \Phi^{2k} + \Phi^{2(k-1)}, \tag{43}$$

where k=0,±1,±2,±3,… .

The identity (41) is a mathematical basis for the ternary mirror-symmetrical summation of two single-digit ternary digits and gives a rule of the carry-over formation (Table 4).

Table 4. Ternary mirror-symmetrical summation a_k+b_k

b_k / a_k	$\bar{1}$	0	1
$\bar{1}$	$\bar{1}1\bar{1}$	$\bar{1}$	0
0	$\bar{1}$	0	1
1	0	1	$1\bar{1}1$

The main peculiarity of Table 4 consists in the rule of summation of two ternary units with equal signs, i.e.

$a_k + b_k$	=	c_k	s_k	c_k
1+1	=	1	$\bar{1}$	1
$\bar{1}+\bar{1}$	=	$\bar{1}$	1	$\bar{1}$

We can see that for the ternary mirror-symmetrical summation of ternary numerals of the same sign, the intermediate sum s_k with opposite sign and the carry-over c_k with the same sign appear. However, the carry-over from the k-th digit is spreading simultaneously to the adjacent two digits, namely to the adjacent left-hand, that is, (k+1)-th digit, and to the adjacent right-hand, that is, (k-1)-th digit.

Table 4 describes an operation of the simplest ternary mirror-symmetrical *adder* called the *single-digit ternary mirror-symmetrical half-adder*. This half- *adder* is a combinative logic circuit that has two ternary inputs a_k and b_k and two ternary outputs s_k and c_k. It operates in accordance with Table 4 (Fig. 12-a).

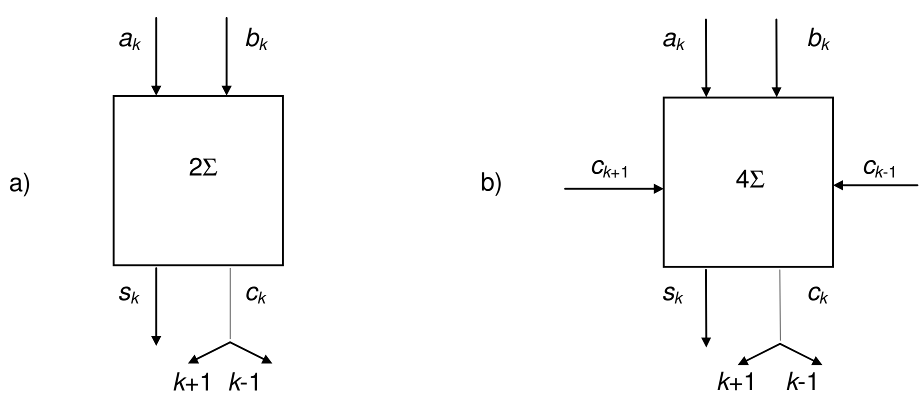

Fig. 12. Mirror-symmetrical single-digit adders: (a) half- adder; (b) full adder

As the carry-over from the k-th digit is spreading to the left-hand and to the right-hand digits, this means that the full mirror-symmetric single-digit *adder* has to have two additional inputs for the carry-overs that come from the (k-1)-th and (k+1)-th digits to the k-th digit. Thus, the full mirror-symmetric single-digit *adder* is a combinative logic circuit that has 4 ternary inputs and 2 ternary outputs (Fig. 12-b). Let us denote by 2Σ the mirror-symmetric single-digit half- *adder* that has 2 inputs and by 4Σ the mirror-symmetric single-digit full *adder* that has 4 inputs.

Now, let us describe the logical operation of the mirror-symmetric full single-digit *adder* of the kind 4Σ. First of all, we note that the number of all possible 4-digit ternary input combinations of the mirror-symmetrical full *adder* in Fig. 12-b is equal to 3^4=81. The values of the output variables s_k and c_k are some discrete functions of the algebraic sum S of the input ternary variables a_k, b_k , c_{k-1}, c_{k+1}, that is,

$$S=a_k+b_k+c_{k-1}+c_{k+1}. \quad (44)$$

The sum (44) takes the values from the set {-4,-3,-2,-1,0,1,2,3,4} The operation rule of the mirror-symmetrical full *adder* of the kind 4Σ (Fig. 12-b) consists in the following. The *adder* forms the output ternary code combinations $c_k s_k$ in accordance with the value of the sum (41) as follows:

$$-4=\bar{1}\bar{1};\ -3=\bar{1}0;\ -2=\bar{1}1;\ -1=0\bar{1};\ 0=00; 1=01; 2=1\bar{1}; 3=10; 4=11.$$

The lower digits of such 2-digit ternary representations are values of the intermediate sums s_k and the higher digit are the values of the carry-over's c_k that are spreading to the neighboring (the left-hand and the right-hand) digits.

Note that the functioning rule of the ternary mirror-symmetrical *adder* in Fig. 12 fully coincides with the functioning rule of the classical ternary-symmetrical *adder*.

2.3.4 Ternary mirror-symmetrical multi-digit adder

The multi-digit combinative mirror-symmetric adder (Fig. 13) that performs the summation of two ($2m$+1)-digit mirror-symmetrical numbers is a combinative logic circuit that consists of ($2m$+1) ternary mirror-symmetric summators of the kind 4Σ (Fig. 12-b).

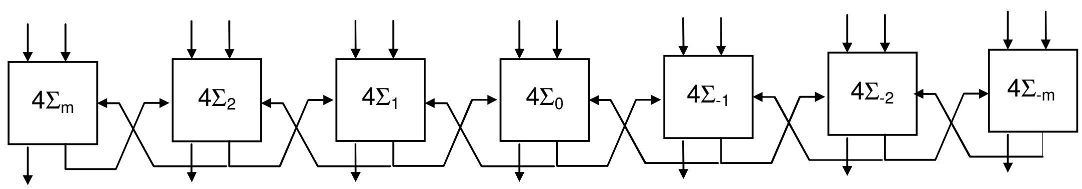

Fig. 13. Ternary mirror-symmetrical multi-digit adder

We can see from Fig. 13 that the main peculiarity of the ternary mirror-symmetric multi-digit *adder* consists in the fact that the carry-over from each digit is spreading symmetrically to the adjacent digits to the left and to the right. Two mirror-symmetrical numbers A and B enter the multi-digit input of the *adder*. The single-digit adder $4\Sigma_0$ separates the *adder* into two parts: the single-digit *adders* $4\Sigma_1, 4\Sigma_2, 4\Sigma_3$ for the highest digits and the single-digit *adders* $4\Sigma_{-1}, 4\Sigma_{-2}, 4\Sigma_{-3}$ for the lowest digits.

Numerical example

Sum up two ternary mirror-symmetric numbers 5+10:

$$\begin{array}{ccccccccc}
5 & = & 0 & 1 & \bar{1} & 1, & \bar{1} & 1 & 0 \\
10 & = & 0 & 1 & 1 & 0, & 1 & 1 & 0 \\
S_1 & = & 0 & \bar{1} & 0 & 1, & 0 & \bar{1} & 0. \\
C_1 & & 1 & \leftrightarrow & 1 & & 1 & \leftrightarrow & 1 \\
\hline
15 & = & 1 & \bar{1} & 1 & 1, & 1 & \bar{1} & 1
\end{array}$$

Note that the symbol $\leftrightarrow$ marks the process of carry-over spreading.

We can see that the addition process for this example consists of two steps. The first step is forming the first multi-digit intermediate sum S_1 and the first multi-digit carry-over C_1 according to Table 4. The second step is the summation of the numbers S_1+C_1 according to Table 4. As for this case the second multi-digit intermediate carry-over $C_1=0$, this means that the summation is over and the sum $S_1+C_1=15$ is the summation result. It is important to emphasize that the summation result

$$15_{10} = 1\bar{1}11.1\bar{1}1 \qquad (45)$$

is represented in mirror-symmetrical form.

As noted above, the possibility of summing up all integers (positive and negative) in the "direct" code is an important advantage of the ternary mirror-symmetrical numeral system (29), that is, we do not use the notions of inverse and additional codes.

Numerical example

Sum up the negative mirror-symmetrical number (-24) and the positive mirror-symmetric number 15:

$$\begin{array}{rccccccc}
-24= & \bar{1} & \bar{1} & 0 & 1, & 0 & \bar{1} & \bar{1} \\
15= & 1 & \bar{1} & 1 & 1, & 1 & \bar{1} & 1 \\
S_1 & 0 & 1 & 1 & \bar{1}, & 1 & 1 & 0 \\
C_1'= & & \downarrow & 1 & \leftrightarrow & 1 & \downarrow & \\
C_1''= & \bar{1} & \leftrightarrow & \bar{1} & & \bar{1} & \leftrightarrow & \bar{1} \\
-9= & \bar{1} & 1 & 1 & \bar{1}, & 1 & 1 & \bar{1}
\end{array}$$

We can see that the summation process consists of two steps. The first step is forming the first multi-digit intermediate sum S_1 and the first multi-digit carry-over $C_1 = C_1' + C_1''$ according to Table 4. The second step is to sum up the numbers $S_1 + C_1' + C_1''$. Here, we use the functioning rule of the ternary mirror-symmetric single-digit summator in Fig. 12-b. As for this case the second multi-digit intermediate carry-over $S_1=0$, this means that the summation is over and the sum $S_1 + C_1' + C_1'' = -9$ is the summation result. It is important to emphasize that the summation result

$$-9_{10} = \bar{1}11\bar{1}.11\bar{1} \qquad (46)$$

is negative number because the ternary mirror-symmetrical representation (46) begins with the negative unit $\bar{1}$. In addition, the summation result (43) is represented in mirror-symmetric form what allows checking the summation process.

2.3.5 Ternary mirror-symmetrical subtraction

The subtraction of two mirror-symmetrical numbers N_1-N_2 transforms to the summation if we represent their difference in the following form:

$$N_1-N_2=N_1+(-N_2). \qquad (47)$$

It follows from (47) that until the subtraction we have to take the *ternary inversion* of the subtrahend N_2 according to (26).

2.3.6 The "swing" phenomenon

Now, let us sum up two equal ternary mirror-symmetrical numbers 5+5:

5	=	0	1	$\bar{1}$	1.	$\bar{1}$	1	0
5	=	0	1	$\bar{1}$	1.	$\bar{1}$	1	0
		0	$\bar{1}$	1	$\bar{1}$.	1	$\bar{1}$	0
			↓	1	↔	1	↓	
		1	↔	1		1	↔	1
			$\bar{1}$	↔	$\bar{1}$	↓		
					$\bar{1}$	↔	$\bar{1}$	
		1	1	0	0.	0	1	1
				$\bar{1}$	↔	$\bar{1}$		
			1	↔	1	↓		
			↓		1	↔	1	
		$\bar{1}$	↔	$\bar{1}$		$\bar{1}$	↔	$\bar{1}$
		0	$\bar{1}$	1	$\bar{1}$.	1	$\bar{1}$	0
				1	↔	1		
			$\bar{1}$	↔	$\bar{1}$	↓		
			↓		$\bar{1}$	↔	$\bar{1}$	
		1	↔	1		1	↔	1

It follows from this example, we have found a special summation case called *swing*. If the summation process goes on, then at some step the process of the formation of carries begins to repeat; this means that the process of the summation becomes infinite. The "swing"-phenomenon is similar to *races* that appear in digit automatons, when the electronic elements are switched.

In order to eliminate the "swing"-phenomenon, we use the following effective "technical" method [7]. The “swing”-phenomenon appears in the ternary mirror-symmetric summator in Fig. 14 because the carry-overs come at the same time from two adjacent single-digits adders. A “technical” solution of this phenomenon is to delay the input signals of the single-digit summators with odd indices $(k = \pm 1, \pm 3, \pm 5, ...)$ by one summation step. For this situation at the first step of the summation only the summators with the even indices $(k = 0, \pm 2, \pm 4, \pm 6, ...)$ operate and they form the intermediate sums and corresponding carry-overs to the single-digit summators with the odd indices. Then, at the second summation step the carry-overs that were formed at the first step are summarized with the corresponding ternary variables of the odd digits of the summable numbers. Thanks to such an approach, the "swing"-phenomenon is eliminated.

Now, let us demonstrate the above method to eliminate the "swing"-phenomenon at the summation of the numbers 5+5:

$$
\begin{array}{lccccccccc}
5_{10} & = & 0 & 1 & \bar{1} & 1. & \bar{1} & 1 & 0 \\
5_{10} & = & 0 & 1 & \bar{1} & 1. & \bar{1} & 1 & 0 \\
\hline
S_1 & = & & \bar{1} & & \bar{1}. & & \bar{1} & \\
 & & & \downarrow & & \downarrow & & \downarrow & \\
C_1' & & & \downarrow & 1 & \leftrightarrow & 1 & \downarrow & \\
C_1'' & & 1 & \leftrightarrow & 1 & & 1 & \leftrightarrow & 1 \\
\hline
10_{10} & = & 1 & \bar{1} & 0 & \bar{1}. & 0 & \bar{1} & 1
\end{array}
$$

The first step of the mirror-symmetrical summation is to summarize all the input ternary numerals with even indices (2,0,-2). The ternary numerals of all digits with odd indices (3,1,-1,-3) are delayed at the first step. The second step is the summation of all the carry-overs, which appear at the first step, with the input ternary numerals of the digits with odd indices. It is important to emphasize that the result of the summation

$$10_{10} = 1\bar{1}0\bar{1}.0\bar{1}1 \tag{48}$$

is a positive number because the ternary representation (48) begins with the positive ternary numeral 1 and in addition the result of the summation (48) is represented in mirror-symmetric form.

An analysis of all the above examples of ternary mirror-symmetric summation shows that both the final result of the summation and all intermediate results are mirror-symmetric numbers, that is, the property of mirror symmetry is an invariant of mirror-symmetrical summation. This means that mirror-symmetrical summation (and subtraction) possesses the important mathematical property of "mirror symmetry" what allows checking the ternary mirror-symmetrical summation and subtraction.

2.3.7 Ternary mirror-symmetrical multiplication and division

The following trivial identity for the golden ratio powers underlies the ternary mirror-symmetrical multiplication:

$$\Phi^{2n} \times \Phi^{2m} = \Phi^{2(n+m)}. \tag{49}$$

The rule of the mirror-symmetric multiplication of two single-digit ternary mirror-symmetric numbers is given in Table 5.

Table 5. Ternary mirror-symmetrical multiplication

b_k \ a_k	$\bar{1}$	0	1
$\bar{1}$	1	0	$\bar{1}$
0	0	0	0
1	$\bar{1}$	1	1

The ternary mirror-symmetrical multiplication is performed in the "direct" code. The general algorithm of the multiplication of two multi-digit mirror-symmetrical numbers is reduced to the formation of the partial products in accordance with Table 5 and their summation in accordance with the rule of the ternary mirror-symmetric addition.

Numerical example

Multiply the negative mirror-symmetric number $-6_{10}=\bar{1}01.0\bar{1}$ by the positive mirror-symmetric number $2_{10}=1\bar{1}.1$:

$$
\begin{array}{ccccccc}
 & \bar{1} & 0 & 1. & 0 & \bar{1} & \\
 & & 1 & \bar{1}. & 1 & & \\
\hline
 & & \bar{1} & 0. & 1 & 0 & \bar{1} \\
 & 1 & 0 & \bar{1}. & 0 & 1 & \\
\bar{1} & 0 & 1 & 0. & \bar{1} & & \\
\hline
\bar{1} & 1 & 0 & \bar{1}. & 0 & 1 & \bar{1}
\end{array}
$$

The multiplication result in this example is formed as the sum of the three partial products. The first partial product $\bar{1}\ 0.\ 1\ 0\ \bar{1}$ is the result of multiplication of the negative mirror-symmetrical multiplier $-6_{10}=\bar{1}01.0\bar{1}$ by the lowest positive ternary numerals 1 of the positive mirror-symmetrical multiplier $2_{10}=1\bar{1}.1$, the second partial product $1\ 0\ \bar{1}.\ 0\ 1$ is the result of the multiplication of the same number $-6_{10}=\bar{1}01.0\bar{1}$ by the middle negative ternary numeral $\bar{1}$ of the number $2_{10}=1\bar{1}.1$, and, finely, the third partial product $\bar{1}\ 0\ 1.\ 0\ \bar{1}$ is the result of the multiplication of the same number $-6_{10}=\bar{1}01.0\bar{1}$ by the higher positive ternary numeral 1 of the number $2_{10}=1\bar{1}.1$.

Note that the product $-12_{10}=\bar{1}\ 1\ 0\ \bar{1},\ 0\ \bar{1}\ \bar{1}$ is represented in the mirror-symmetric form! Because its higher digit is a negative ternary numeral $\bar{1}$, it follows from here that the product is a negative mirror-symmetrical number.

As for the mirror-symmetrical division, it generally similar to the division rule in the classical ternary symmetrical numeral system (22). The detailed description of the ternary mirror-symmetrical division is given in the article [7].

3 Matrix and Pipeline Mirror-symmetrical Adder and Multiplier

3.1 Matrix mirror-symmetrical adder

It is well known that the digital signal processors put forth high demands to the speed of arithmetical devices. The different special structures (matrix, pipeline, etc.) are elaborated for this purpose. We can show that the ternary mirror-symmetrical arithmetic contains in itself the interesting possibilities for designing fast arithmetical devices for signal processors.

Consider now the matrix multi-digit ternary mirror-symmetric adder (Fig. 14). Each cell of the matrix adder in Fig. 14 is full ternary-symmetrical single-digit adder of the kind $4\sum$, which have the 4 inputs and 2 outputs (see Fig. 12-b). The matrix adder in Fig. 14 consists of the 21 single-digit full adders $4\sum$, which are arranged in the form of the 7×3-matrix. Each ternary single-digit adder has designation $4\Sigma_3^2$, where the number 4 means that the adder $4\Sigma_i^k$ has 4 ternary inputs, the indexes *i* and *k* in the adder $4\Sigma_i^k$ mean that the adder $4\Sigma_i^k$ refers to the *i*-th digit of the ternary mirror-symmetric representation (27) and the adder are placed in the *k*-th row of the matrix adder in Fig. 14.

The inputs of the single-digit adders

$$4\Sigma_3^1, 4\Sigma_2^1, 4\Sigma_1^1, \Sigma_0^1, 4\Sigma_{-1}^1, 4\Sigma_{-2}^1, 4\Sigma_{-3}^1$$

of the first row form the multi-digit input of the matrix ternary-symmetrical adder in Fig. 14. The output of the intermediate sum of each single-digit adder is connected to the corresponding input of the next single-digit adder of the same column.

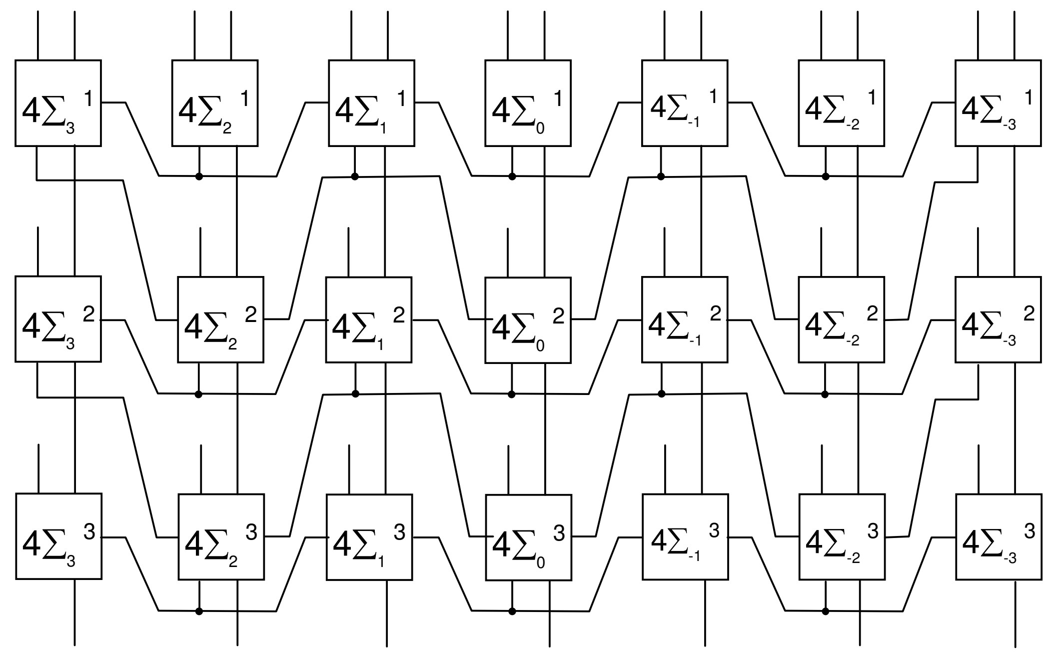

Fig. 14. Matrix ternary mirror-symmetrical adder

The outputs of the intermediate sum of the single-digit adders

$$4\Sigma_3^1, 4\Sigma_2^1, 4\Sigma_1^1, \Sigma_0^1, 4\Sigma_{-1}^1, 4\Sigma_{-2}^1, 4\Sigma_{-3}^1$$

of the last row form the multi-digit output of the matrix mirror-symmetric adder.

The main peculiarity of the matrix mirror-symmetric adder in Fig. 14 consists in a special designing the connections between the carry outputs of the single-digit adders and the inputs of the neighboring single-digit adders. The carry outputs of all single-digit adders with the even lower indices (2, 0, -2) are connected to the corresponding inputs of the adjacent single-digit adders, which are placed in the same row, but the carry outputs of all the single-digit adders with the odd lower indices (3, 1, -1, -3) are connected with the corresponding inputs of the adjacent single-digit adders, which are placed in the lower row. Note that such organization of the carry connections allows eliminating the above "swing" phenomenon.

Consider the operation of the matrix mirror-symmetric adder on the example of the addition of two equal ternary mirror-symmetric numbers:

$A = 0\ 1\ 1\ 1,\ 1\ 1\ 0$ and $B = 0\ 1\ 1\ 1,\ 1\ 1\ 0.$

The addition is fulfilled in 2 stages. Each stage is fulfilled by means of one row of the single-digit adders and consists of two steps.

The first stage

In accordance with Fig. 14, the first step of the first stage consists in the following. The single-digit adders of the first row with the even lower indices ($4\Sigma_2^1$, $4\Sigma_0^1$, $4\Sigma_{-2}^1$) form the intermediate sums, which enter the inputs of the second row adders, and the carries, which enter the corresponding inputs of the single-digit adders with the *odd* lower indices of the first row ($4\Sigma_3^1$, $4\Sigma_1^1$, $4\Sigma_{-1}^1$, $4\Sigma_{-3}^1$). Such transformation of the code information can be represented in the following form:

$$
\begin{array}{ccccccc}
0 & 1 & 1 & 1. & 1 & 1 & 0 \\
0 & 1 & 1 & 1. & 1 & 1 & 0 \\
\hline
 & \bar{1} & & \bar{1} & & \bar{1} & \\
 & \downarrow & 1 & \leftrightarrow & 1 & \downarrow & \\
1 & \leftrightarrow & 1 & & 1 & \leftrightarrow & 1
\end{array}
$$

Hence, the first step is the formation of the intermediate sums and the carries on the outputs of the single-digit adders with the *even* lower indices (2, 0, -2).

At the second step of the first stage the single-digit adders with the *odd* lower indices (3, 1, -1, -3) go into action. In accordance with the entered carries, they form the intermediate sums and the carries, entering the single-digit adders of the lower row, that is,

$$
\begin{array}{ccccccc}
0 & 1 & 1 & 1. & 1 & 1 & 0 \\
0 & 1 & 1 & 1. & 1 & 1 & 0 \\
\hline
 & \bar{1} & & \bar{1} & & \bar{1} & \\
 & \downarrow & 1 & \leftrightarrow & 1 & \downarrow & \\
1 & \leftrightarrow & 1 & & 1 & \leftrightarrow & 1 \\
\hline
1 & \bar{1} & 1 & \bar{1}. & 1 & \bar{1} & 1 \\
 & 1 & \leftrightarrow & 1 & \downarrow & & \\
 & & & 1 & \leftrightarrow & 1 &
\end{array}
$$

The first stage is over. We can see that the results of the first stage are some intermediate sum and some carries, entering the adders of the lower row.

The second stage

The single-digit adders of the second row with the *even* lower indices ($4\Sigma_2^2$, $4\Sigma_0^2$, $4\Sigma_{-2}^2$) form the intermediate sums, entering the corresponding inputs of the lower row adders and the carries, entering the corresponding inputs of the same row adders with the odd lower indices ($4\Sigma_3^2$, $4\Sigma_1^2$, $4\Sigma_{-1}^2$, $4\Sigma_{-3}^2$), that is,

$$\begin{array}{ccccccc}
1 & \bar{1} & 1 & \bar{1}. & 1 & \bar{1} & 1 \\
 & 1 & \leftrightarrow & 1 & \downarrow & & \\
 & & & 1 & \leftrightarrow & 1 & \\
\hline
1 & 0 & 1 & 1. & 1 & 0 & 1
\end{array}$$

Because all carries, which are formed at this stage, became equal to 0, this means that the addition is over at the second stage (this is true only for the considered case). The obtained sum enters the inputs of the lower row adders $4\Sigma_3^3 - 4\Sigma_{-3}^3$ and then appears on the output of the matrix adder.

3.2 The pipeline mirror-symmetrical adder

There are two ways for the extension of the functional possibilities of the matrix mirror-symmetrical adder in Fig. 14. If we set the ternary registers, which consist of the flip-flop-flaps (see Fig. 12-b) between the adjacent rows of the single-digit adders, then the above matrix adder turns into the *pipeline ternary mirror-symmetrical adder*. In fact, the code information from the preceding rows of the single-digit adders is memorized in the corresponding registers and the preceding row of the adders becomes is ready for further processing. Then, the adders of the lower row process the code information, entering the lower row of the single-digit adders, and simultaneously the top row of the single-digit adders starts to process the new input code information. This means that since the given moment we will get the sums of the numbers $A_1 + B_1$, $A_2 + B_2$, …, $A_n + B_n$, entering the adder input during the time period $2\Delta\tau$, where $\Delta\tau$ is the delay time of the single-digit adder.

3.3 The pipeline mirror-symmetrical multiplier

The other possibility to extend functional possibilities of the pipeline adder consists in the following. We can see in Fig. 14 that each single-digit adder of the lower rows has a "free" input. We can use these inputs as the new multi-digit inputs of the pipeline adder. By using these multi-digit inputs, we can turn the pipeline adder into the pipeline multiplier. In this case the mirror-symmetrical multiplication of two mirror-symmetric numbers $A(1)\times B(1)$ is performed in the following manner. The first row of the single-digit adders summarizes the first two partial products $P_1^1 + P_2^1$. This code information enters the second row of the single-digit adders. If we send the 3-rd partial product P_3^1 to the "free" input of the second row, we will get the sum $P_1^1 + P_2^1 + P_3^1$ on the outputs of the second row. In this case the first row starts to sum the first two partial products of the next pair of multiplied numbers $A(2)\times B(2)$. The "free" input of the 3-rd row is used to accept the next partial product P_4^1 of the first pair of the multiplied numbers $A(1)\times B(1)$, etc. We can see that the pipeline adder in Fig. 14 allows to multiply many mirror-symmetric numbers in the pipeline regime. In this connection the multiplication speed is determined by the time $2\Delta\tau$, where $\Delta\tau$ is the delay time of the single-digit adder.

2 Conclusions

The main conclusions, arising from the above reasoning's, are as follows:

1. The binary numeral system with the irrational base of Φ (*Bergman's system*) and the ternary mirror-symmetrical numeral system with the base of Φ^2 (*Stakhov's system*) are very important for future applications in mathematics, theoretical physics and computer science.

2. The historical importance of *Bergman's system* is correction of "strategic mistake" in the development of mathematics, by recovering in modern mathematics Pythagorean MATEM of *harmonics*, lost by mathematics in the process of its historical development. The same could be said for the *Stakhov's system.*
3. A conceptions of *Bergman's system* opens up a new direction in the development of number theory (the *"golden" number theory*), and consequently, leads to new fundamental results in this field, in particular, to new unexpected properties of natural numbers (*Z-* and *D-properties, Φ-, F-, L-codes* and so on) .
4. *Bergman's system* is possibly the most important mathematical discovery in the field of positional numeral systems after the invention of the positional principle of number representation (Babylon, 2000 B.C.E.) and also decimal and binary systems. The importance of *Bergman's system* for the development of numeral systems, mathematics and computer science can be compared with the introduction of irrational numbers by Pythagoreans in Ancient Greece.
5. We can say the same on *Stakhov's ternary mirror-symmetrical numeral system* with the irrational base of $\Phi^2 = \frac{3+\sqrt{5}}{2}$. This numeral system follows from *Bergman's system* and *Brousentsov ternary principle* and combines in itself all advantages of *Bergman's system* and classical *ternary numeral system. Stakhov's system* has the following useful properties:

   5.1 All arithmetical operations are performed in *"direct"* form (without using *additional* and *inverse* codes). All integers (positive and negative) are represented in *mirror-symmetrical form.* This means that at the representation of integers, the 0^{th} digit divides the *ternary mirror-symmetrical representation* into two mirror-symmetric parts. At the increasing of number its ternary mirror-symmetric representation is expanded symmetrically to the left and to the right relative to 0^{th} digit. This unique mathematical property generates very simple method of error detection in mirror-symmetrical computers and processors.

   5.2 It is proved that the *mirror-symmetric property* is invariant relative to arithmetical operations, that is, the results of all arithmetical operations always have *mirror-symmetrical form*. This means that the mirror-symmetrical numeral system and arithmetic can be used for designing of specialized self-checking mirror-symmetrical computers and processors.

   5.3 The *ternary mirror-symmetrical numeral system* is possibly the final stage in the long historical development of the concept of *ternary numeral systems*, because in the *ternary mirror-symmetrical numeral system* two scientific problems, the *sign problem and representation of negative numbers* and *problem of error detection*, based on the *principle of mirror symmetry*, are solving simultaneously. The author is ready to offer consulting services for any electronic company with advanced technology, which can be interested in the technical implementation of the ternary mirror-symmetrical processors and computers on this basis.

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Competing Interests

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