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On Statistical Continuity of Functions

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

The idea of statistical convergence of sequences in Hausdorff topological spaces was introduced and studied to some extend by Di Maio and Kocinac in [1]. In this paper we consider the concept of statistical continuity of functions and give a characterization of them by using statistical convergent sequences in first countable Hausdorff topological spaces.

Keywords: Statistical convergent sequence; statistical convergent function; first countable Hausdorff space.

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1 Introduction

In general, it is known that the notion of statistical convergence of sequences of real numbers was introduced by H. Fast in [2] and H. Steinhaus in [3] and it is based on the notion of asymptotic density of a set $A \subset N$ [4,5,6]. However, as stated in [1], the first idea of statistical convergence appeared in the first edition of the celebrated monograph [7] of Zygmund but under a different name as almost convergence. It should be expressed the notion of statistical convergence has been considered, in some other contexts, by several

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people like R.A. Bernstein, Z. Frolik, etc. We come across to applications of the statistical convergence concept in different fields of mathematics such as summability theory [8,9,10,11], number theory [12], trigonometric series [7], probability theory [13], measure theory [14], optimization [15] and approximation theory [16]. The statistical convergence was generalized to sequences in metric spaces (see, for instance, [17]). Di Maio and Kocinac [1] introduced and studied statistical convergence in Hausdorff topological spaces and uniform spaces and offered some applications to selection principles theory, function spaces and hyperspaces. After introducing by Fast [2], it has been a very rapid investigation concerning the notion of statistical convergence of sequences of real numbers as in the papers [8,10,18,19,20]. After Freadman and Sember [9], Kolk [18], taking in Fast's definition of statistically convergent sequence and Fridy's definition of statistically Cauchy sequence an arbitrary non-negative regular matrix A instead of Cesaro C_1, introduced the notions of A-statistically convergent and A-statistically Cauchy sequences in normed spaces. Independly Maddox [19] introduced the statistical convergence in locally convex spaces. Statistical limits of measurable functions were considered by F. Moricz in [21] to some extent. Counting as recent examples of studies on similar topics; A. Boccuto et al presented some different types of ideal convergence/divergence and of ideal continuity for Riesz space-valued functions, and prove some basic properties and comparison results in [22]. B. T. Bilalov and T. Y. Nazarova studied on the statistical type convergence and fundamentality in metric spaces in their work [23]. B.T. Bilalov and S.R. Sadigova introduced the concept of statistical continuity in [24]. Then B.T. Bilalov and T. Y. Nazarova considered the concept of statistical convergence in metric spaces proving its equivalence to the statistical fundamentality in complete metric spaces in [25]. In the same paper after introducing the concept of p-strong convergence they proved its equivalence to the statistical convergence and gave Tauberian theorems concerning statistical convergence in metric spaces. They also considered statistical convergence in Lebesgue spaces in [26] and gave a criterion for statistical convergence.

It also should be expressed that the concept of statistical convergence in topological spaces was introduced by Di Maio and Kocinac and studied the concept to some extend [1]. Inspired by [1], we introduce the definition of the statistical continuous function concept and then give a characterization of statistical continuity of functions by means of using statistical convergent sequences in Hausdorff topological spaces.

2 Definitions and Basic Properties

We shall start with the remembrance of some basic definitions, notations and auxiliary results concerning the concept of statistical convergence.

Definition 2.1. Let $A \subset N = \{1,2,3,...\}$ and $n \in N$. Put $A(n) = \{a \in A: \ a \le n\}$. If there exist $\liminf_{n\to\infty} \frac{|A(n)|}{n}$, it will be called the asymptotic density of the set A and denoted by $\delta_1(A)$, where $|A|$ denotes the cardinality of any set $A \subset N$. If we have the limit $\lim_{n\to\infty} \frac{|A(n)|}{n}$, then this limit value is called the natural density of the set A and denoted by $\delta(A)$. Therefore, if the set A has the the natural density, we have $\delta(A) = \delta_1(A) = \lim_{n\to\infty} \frac{|A(n)|}{n}$. Note that the density values, if they exists, are in the interval [0,1]. Obiviously we have $\delta(A) = 0$ provided that A is a finite set of positive integers.

Let's give a definition concerning the notion of a statistically convergent sequence introduced by Fast [2].

Definition 2.2. The sequence $(x_n)_{n\in N}$ of real numbers is said to convergence statistically to the real numbers x and denoted by $\lim_{n\to\infty}(stat - x_n) = x$ or briefly $x_n \xrightarrow{st} x$ if for each $\varepsilon > 0$ we have $\delta(A_\varepsilon) = 0$, where $A_\varepsilon = \{n \in N : |x_n - x| \geq \varepsilon\}$. It is clear that $\delta(A_\varepsilon) = 0$ equals to $\delta(A_\varepsilon{}^c) = 1$, where $A_\varepsilon{}^c = \{n \in N : \; |x_n - x| < \varepsilon \;\} = N - A_\varepsilon$.

Another well-known definition of a statistically convergent sequence (x_n) is such as following: We have $x_n \xrightarrow{st} x$ if $\lim_{n\to\infty}\frac{1}{n}\left|\{k \leq n : |x_k - x| \geq \varepsilon\}\right| = 0$.

3 The Main Results

We start by giving the definition of topological statistically convergence concept for a sequence in a topological space as done by Di Maio and Kocinac in [1].

Definition 3.1. Let X be a non empty set and (X, τ) be any topological space.

The sequence $(x_n)_{n\in N}$ of *X* is said to converge topological statistically to the element x in the topological space *X* if for each neighbourhood *U* in the $N(x)$ (where $N(x)$ is a collection of the neighbourhoods of element x) we have $\delta(A_U) = 0$, where $A_U = \{n \in N : x_n \notin U \;\}$. We denote this case by $\lim_{n\to\infty}(\tau - stat\; x_n) = x$ or briefly $x_n \xrightarrow{\tau - st} x$. When $\lim_{n\to\infty} x_n = x$, for each $\varepsilon > 0$, there exists such a n_0 that for each $n_0 \leq n$, $n \in N$ we have $|x_n - x| < \varepsilon$, following Proposition 3.2. is trivial due to the definition of the topological statistically convergence.

Proposition 3.2. If $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty}(\tau - stat\; x_n) = x$ holds.

The conclusion contained in following Lemma 3.3 can be also found as a statement in [1].

Lemma 3.3. [1] Let (X, τ) be a Hausdorff topological space, then topological statistically limit is unique.

Proof. Take a sequence $(x_n) \subset X$ and suppose we have $\lim_{n\to\infty}(\tau - stat\; x_n) = x$ and $\lim_{n\to\infty}(\tau - stat\; x_n) = y$. We should show the equality $x = y$. For the contradiction, suppose $x \neq y$. Since *X* Hausdorff topological space, there exist $U \in N(x)$ ve $V \in N(y)$ such that $U \cap V = \varnothing$. By considering, $A_U{}^c = \{n \in N : x_n \in U \;\}$, $A_V{}^c = \{n \in N : \; x_n \in V \;\}$ and $\delta(A_U{}^c) = \delta(A_V{}^c) = 1$, we have $A_U{}^c \cap A_V{}^c \neq \varnothing$. Then, there exists at least an $k \in N$ such that $x_k \in U \cap V$, but it contradicts with the assumption. Hence $x = y$.

Following Theorem 3. 4. is a generalisation of a well-known result in the case of statistical convergence for sequences of real numbers to the first countable Hausdorff topological spaces. For the sake of completeness, it is presented to the readers a clearly understandable proof of Theorem 3.4. which belongs to Di Maio and Kocinac in [1].

Theorem 3.4. [1] Let (X,τ) a first countable Hausdorff topological space, $(x_n)\subset X$ and $x\in X$. Then, $x_n \xrightarrow{\tau - st} x$ if and only if there exists a set $K\subseteq N$ such that $\delta(K)=1$ and $x_{k_n}\to x$

Proof. Since X first countable topological space we have a decreasing collection of the neighbourhoods of the element x, that is a local base, $U_1\supseteq U_2\supseteq U_3\supseteq ...\supseteq U_n$.

Let $K_j=\{n\in N:\ x_n\in U_j\ \}$ and $j=1,2,3,...,n$. Then we have $\delta(K_j)=1$ such that $K_{j+1}\subseteq K_j\subseteq ...\subseteq K_2\subseteq K_1$. Take any $V_1\in K_1$. Since $\delta(K_2)=1$, we have $\lim_{n\to\infty}\frac{1}{n}K_2(n)$ =1. Take $\varepsilon=\frac{1}{2}$, then for $\exists n_0 : n_0\le n$ we have $\frac{1}{2}=1-\frac{1}{2}<\frac{1}{n}K_2(n)<1+\frac{1}{2}$
$\max\{n_0,V_1\ \}=N<V_2$ and $V_2\in K_2$ such that, for $V_2\le n$ we get $\frac{1}{2}<\frac{1}{n}K_2(n)$. If we continue to apply the same thing for $\varepsilon=\frac{1}{3}$ and K_3, we have $\frac{1}{n}K_3(n)>\frac{2}{3}$ for $V_2<V_3$ and $V_3\le n$. By this way, we obtain, with induction, positive integers such that $V_1<V_2<V_3<...<V_j<...$. Now $V_j\in K_j\ (j=1,2,3,...)$ and $\frac{1}{n}\mathrm{K}_j(n)>\frac{j-1}{j}$ for each $n\ge V_j$.

Let construct a set K such as $K=([V_0,V_1)\cap N)\cup([V_1,V_2)\cap K_1)\cup...$. We can write it as $K=\bigcup_{j=0}^{\infty}[V_j,V_{j+1})\cap K_j$.

Since $V_j\le n<V_{j+1}$ and $\frac{1}{n}K(n)\ge\frac{1}{n}K_j(n)>\frac{j-1}{j}$ for each *n*, we have $\delta(K)=1$.

Now we should show that for any $U\in N(x)$ we have $\exists k_0\in K$ such that $x_k\in U$ for each $k_0\le k$. Let U_j be a set such that $U_j\subseteq U$ and $V_j\le n$ for $n\in K$. Then there exists at least a number i such that $i\ge j$ and $V_i\le n\le V_{i+1}$. From the definition of the set *K*, we know $n\in K_i$. Hence we have $x_n\in U_i\subseteq U_j\subseteq U$. Therefore we see, if $n\in K$, there exists a V_j such that $V_j\le n$ and $x_n\in U$. As a result we have $(x_k)_{k\in K}\to x$.

For the opposite side, let $K=\{k_1<k_2<...<k_n<...\ \}\subseteq N$, $\delta(K)=1$ and $(x_k)_{k\in K}\to x$. For each $U\in N(x)$, we have some $n_0\in N$ such that $x_{k_n}\in U$ for every $n_0\le n$. Let $A_U=\{n\in N: x_n\notin U\ \}$ then we see $A_U\subseteq N-\{k_{n_0+1},k_{n_0+2},...\ \}$. Since $\delta($ $N-\{k_{n_0+1},k_{n_0+2},...\ \}$)=0 we have $\delta(A_U)=0$ which completes the proof.

As can be inferenced from the below proof, when (X,τ) is not first countable Hausdorff topological space Theorem 3.4. may not hold.

Theorem 3.5. Let (X,τ_1) and (Y,τ_2) be two first countable Hausdorff topological spaces and $f:(X,\tau_1)\to(Y,\tau_2)$ be a given function. Then the following are equivalent.

- **a)** f is continuous.
- **b)** If $x_n \xrightarrow{\tau-st} x$ then $f(x_n)\xrightarrow{\tau-st} f(x)$.
- **c)** If $x_n \to x$ then $f(x_n)\xrightarrow{\tau-st} f(x)$.

Proof: **a$\Rightarrow$b:** Let $x_n \xrightarrow{\tau-st} x$, then the conditions $\exists K\subseteq N:\delta(K)=1$ and $x_{k_n}\to x$ are satisfied from the Theorem3.4. Since f is continuous, the convergence $f(x_{k_n})\to f(x)$ implies the convergence $f(x_n)\xrightarrow{\tau-st} f(x)$.

b$\Rightarrow$c: Let $x_n\to x$ be given. Then $x_n \xrightarrow{\tau-st} x$ satisfies. From the hypothesis in b) the convergence $f(x_n)\xrightarrow{\tau-st} f(x)$ satisfies.

c $\Rightarrow$a: To show the continuity of the function $f:(X,\tau_1)\to(Y,\tau_2)$, we will use closed sets in the space *Y*. (We denote the closure of any set *U* by $\overline{U}$). Let *U* be a closed set in the topological space *Y*. It is enough to show that the set $f^{-1}(U)$ is closed in the topological space *X*. Take any $x\in \overline{f^{-1}(U)}$. Since *X* first countable topological space there exists a sequence (x_n) in *X* such that $x_n\to x$. Then from the hypothesis we get $f(x_n)\xrightarrow{\tau-st} f(x)$. The conditions $\exists K\subseteq N:\delta(K)=1$ and $f(x_{k_n})\to f(x)$ are satisfied from the Theorem3.4. By $f(x_{k_n})\in U$ and $f(x)\in\overline{U}$ =*U* we have $x\in f^{-1}(U)$ and so $\overline{f^{-1}(U)}\subseteq f^{-1}(U)$. Since the set $f^{-1}(U)$ is closed in *X* we see the function *f* is continuous.

Definition 3.6. Let (X,τ_1) and (Y,τ_2) be two first countable Hausdorff topological spaces and $f:(X,\tau_1)\to(Y,\tau_2)$ be a given function. f is said to be " $\tau_1-\tau_2$ statistical continuous" if the convergence $x_n \xrightarrow{\tau-st} x$ implies the convergence $f(x_n)\to f(x)$ for every $x\in X$.

Theorem 3.7. Let (X,τ_1) and (Y,τ_2) be two first countable Hausdorff topological spaces. Every $\tau_1-\tau_2$ statistical continuous function $f:(X,\tau_1)\to(Y,\tau_2)$ is continuous.

Proof: Since continuity and sequential continuity are equivalent in first countable Hausdorff topological spaces it is enough to show *f* is sequentially continuous**.** When $x_n \longrightarrow x$ we have $x_n \xrightarrow{\tau - st} x$, so from the definition, the convergence $f(x_n) \longrightarrow f(x)$ satisfies which means f sequential continuous.

The converse of Theorem 3.7 does not hold. There may be any continuous function $f:(X,\tau_1) \rightarrow (Y,\tau_2)$ which is not $\tau_1 - \tau_2$ statistical continuous. For example; the identity function $I: R \rightarrow R,\ I(x) = x$ continuous according to usual topologies for the both sides, but it is not $\tau_1 - \tau_2$ statistical continuous because, when $x_n \xrightarrow{\tau - st} x$, the convergence $x_n \longrightarrow x$ does not satisfy. Since setting a theory of continuity of functions on limit theory is very natural and Theorem 3.7 implies that statistical continuity is stronger than simple continuity, this result may stimulate a promising research area mentioned in the introduction part.

4 Conclusion

In this paper, investigating the relationship between the statistical continuity and the simple continuity, we firstly consider the concept of statistical continuity of functions and then give a characterization of them by using statistical convergent sequences in first countable Hausdorff topological spaces. The difference between the results in this paper and those existing in the literature so far is that those well-known latter ones are satisfied in some specific spaces such as measurable spaces or metric spaces. As to this study, it implies that statistical continuity is stronger than simple continuity for general first countable Hausdorff topological spaces which is an important contribution to the related research area. This result can be considered as a small beginning in the theory of statistical continuity of functions concerning limit theory.

Competing Interests

Author has declared that no competing interests exist.

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