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The Kumaraswamy Exponentiated U-Quadratic Distribution: Properties and Application

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Authors' contributions

This work was carried out in collaboration between all authors. Authors MM and AMY designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. Authors MM and IM managed the analyses of the study. All authors read and approved the final manuscript.

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ABSTRACT

In this paper, a new lifetime model called Kumaraswamy exponentiated U-quadratic (KwEUq) distribution is proposed. Several mathematical and statistical properties are derived and studied such as the explicit form of the quantile function, moments, moment generating function, order statistics, probability weighted moments, Shannon entropy and Renyi entropy. We also found that the usual maximum likelihood estimates (MLEs) fail to hold for the KwEUq distribution. Two alternative methods are suggested for the parameter estimation of the KwEUq, the alternative maximum likelihood estimation (AMLE) and modified maximum likelihood estimation (MMLE). Simulation studies were conducted to assess the finite sample behavior of the AMLEs and MMLEs. Finally, we provide application of the KwEUq for illustration purposes.

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1 INTRODUCTION

In statistical modeling, problems that occur in practical applications encountered researchers to propose new models, so that lifetime data set can be investigated better. In contrast, there is a high need to introduce useful models to analyze the real-life phenomenon which arise in physics, medical science, computer science, industry, communication, biology, engineering, and public health, among others. For example, the well-known classical distributions such as exponential, Weibull are unable to show wide flexibility in modeling data with U-shape or bimodal density.

The method of generalization of distributions by the Kumaraswamy-G generator proposed by [1] is one of the preferable techniques in distribution theory and was considered by many authors in recent years. For instance, the Kumaraswamy-Weibull distribution [2], Kumaraswamy binomial [3], Kumaraswamy generalized gamma [4], Kumaraswamy-Gumbel [5], Kumaraswamy generalized half-normal [6], Kumaraswamy log-logistic [7], Kumaraswamy-Birnbaum-Saunders [8], Kumaraswamy inverse Weibull [9], Kumaraswamy double inverse exponential [10], Kumaraswamy power series [11], Kumaraswamy-Pareto [12], Kumaraswamy generalized linear failure rate [13], Kumaraswamy exponentiated Pareto [14], Kumaraswamy quasi-Lindley [15], Kumaraswamy generalized Pareto [16], Kumaraswamy-Burr XII distribution [17], Kumaraswamy generalized exponentiated Pareto [18], Kumaraswamy generalized Lomax [19], Kumaraswamy geometric [20], Kumaraswamy half-Cauchy distrbution [21], Kumaraswamy generalized Rayleigh [22], Kumaraswamy-Dagum distribution [23], Kumaraswamy inverse Rayleigh [24], Kumaraswamy inverse exponential [25], Kumaraswamy - Kumaraswamy [26], Kumaraswamy transmuted exponentiated modified Weibull [27], Kumaraswamy odd log-logistic [28], Kumaraswamy skew-normal [29].

The Kumaraswamy-G generator of distributions is defined according to [1] as follows.

Definition 1.1. Let $G(x)$ be any base line cumulative distribution function and $g(x)$ be the density function corresponding to $G(x)$. Let $\lambda,\ \delta > 0$, then, the cumulative distribution function (cdf) and density function of the Kumaraswamy-G generator of distributions satisfy the relationship

$$F^{*}(x) = 1 - [1 - G(x)^{\lambda}]^{\delta}. \quad x \in \mathbb{R} \qquad (1.1)$$

and

$$f^{*}(x) = \delta\lambda g(x)G(x)^{\lambda-1}[1 - G(x)^{\lambda}]^{\delta-1}, \quad x \in \mathbb{R} \qquad (1.2)$$

respectively.

In this case, let the baseline distribution $G(x)$ be the cumulative distribution function of the exponentiate U-quadratic (EUq) distribution proposed by [30], given by

$$G(x) = \left(\frac{\alpha}{3}\right)^{\theta}((x-\beta)^{3}+(\beta-a)^{3})^{\theta}, \quad x \in [a, b] \quad (1.3)$$

where $\theta > 0$ is the shape parameter, a the lower limit, b the upper limit and $a \in (-\infty, \infty)$, $b \in (a, \infty)$, $\alpha \in (0, \infty)$, and $\beta \in (-\infty, \infty)$ such that $\alpha = \frac{12}{(b-a)^{3}}$ and $\beta = \frac{b+a}{2}$. The density function of EUq is

$$g(x) = \theta\alpha^{\theta}3^{1-\theta}(x-\beta)^{2}((x-\beta)^{3}+(\beta-a)^{3})^{\theta-1}, \quad x \in [a, b] \qquad (1.4)$$

The main objective of this paper is to propose a new probability model called Kumaraswamy exponentiated U-quadratic (KwEUq) distribution and study some of its important mathematical and statistical properties. In addition, to investigate the parameters estimation and its application to real data.

The rest of the paper is organized as follows. In section 2, the KwEUq distribution and some important properties are discussed. In section 3, the parameters estimation via alternative maximum likelihood and modified maximum likelihood estimation methods are presented. Application of the new model to a real data set is provided in section 4. Conclusions are given in section 5.

2 THE KWEUQ MODEL

The cumulative distribution of the kumaraswamy exponentiated U-quadratic (KwEUq) distribution can be defined by substituting (1.3) into equation (1.1), thus

$$F(x) = 1 - \left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3 + (\beta-a)^3)^{\theta\lambda}\right]^{\delta}, \quad x \in [a,b] \tag{2.1}$$

and the corresponding density function is

$$f(x) = \delta\lambda\theta\alpha^{\theta\lambda}3^{1-\theta\lambda}(x-\beta)^2((x-\beta)^3 + (\beta-a)^3)^{\theta\lambda-1}\left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3 + (\beta-a)^3)^{\theta\lambda}\right]^{\delta-1}, \quad x \in (a,b) \tag{2.2}$$

where θ, $\lambda > 0$ and $\delta > 0$ are the shape parameters. The survival function $s(x)$ and the hazard rate function $h(x)$ of the KwEUq distribution are respectively given by

$$s(x) = \left(1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3 + (\beta-a)^3)^{\theta\lambda}\right)^{\delta}, \quad x \in [a,b] \tag{2.3}$$

and

$$h(x) = \frac{\delta\lambda\theta\alpha^{\theta\lambda}3^{1-\theta\lambda}(x-\beta)^2((x-\beta)^3 + (\beta-a)^3)^{\theta\lambda-1}\left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3 + (\beta-a)^3)^{\theta\lambda}\right]^{\delta-1}}{\left(1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3 + (\beta-a)^3)^{\theta\lambda}\right)^{\delta}}. \tag{2.4}$$

Figures 1, 2 and 3 display the plot of the cumulative distribution function, probability density function and hazard rate function of the KwEUq distribution for some parameter values.

Special Cases :

The special cases of the KwEUq distribution for some selected parameters values are:

1. When $\delta = 1$, we obtain Exponentiated U-quadratic (EUq) distribution with power parameter θ, λ.
2. If we take $\delta = 1$ and $\lambda = 1$, we also obtain Exponentiated U-quadratic (EUq) distribution but with power parameter θ [30].
3. Setting $\theta = 1$, we get Kumaraswamy U-quadratic (KwUq) distribution.
4. If $\theta = 1, \lambda = 1$ and $\delta = 1$, we get U-quadratic (Uq) distribution.

2.1 Quantile and Moments

The quantile function of the KwEUq distribution could be obtained by inverting (2.1) and could be used to determine some measures such as median and other percentiles. The quantile function can also be used to generate a random data which follow according to the KwEUq. The quantile function of KwEUq distribution is

$$\Omega(p) = \left[\frac{(1-(1-p)^{1/\delta})^{1/\theta\lambda}}{\left(\frac{\alpha}{3}\right)} - (\beta-a)^3\right]^{1/3} + \beta. \tag{2.5}$$

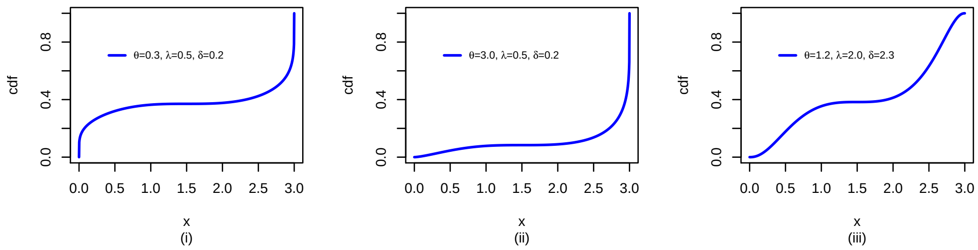

Fig. 1. Plots of the cumulative distribution function of the KwEUq distribution for some different values of parameter.

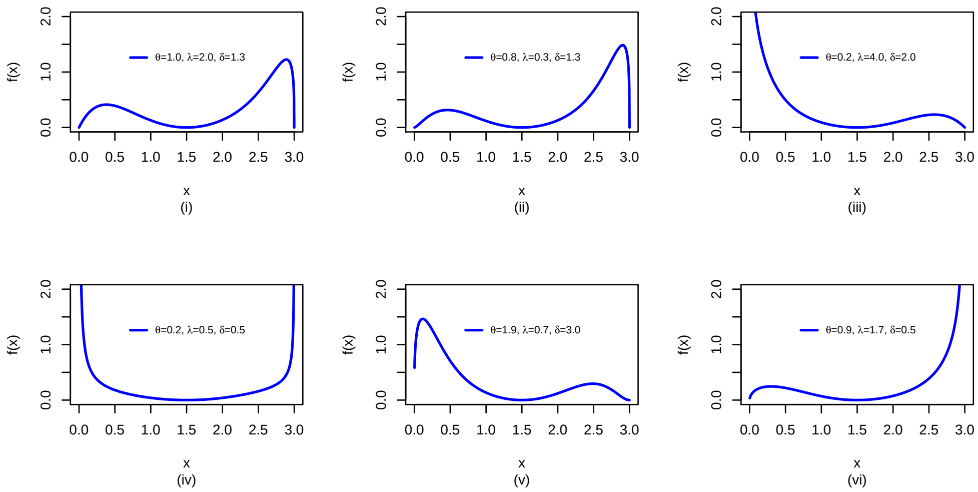

Fig. 2. Plots of the probability density function of the KwEUq distribution for some parameter values.

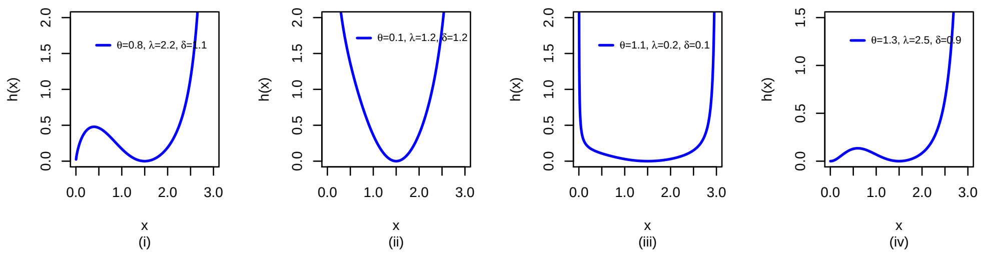

Fig. 3. Plots of the hazard function of KwEUq distribution for some parameter values.

Proposition 2.1. *Let* $Y \sim U(0,1)$, *then* $X = \sqrt[3]{\frac{(1-(1-Y)^{1/\delta})^{1/\theta\lambda}}{(\frac{\alpha}{3})}} - (\beta - a)^3 + \beta$ *is the random variable with*

$KwEUq(\theta,\lambda,\delta,a,b)$, *where* $U(0,1)$ *is uniform distribution.*

The following lemma 2.1 is very useful in computations of several important properties of the KwEUq distribution.

Lemma 2.1. *For* w_1, w_2, w_3 *and* $w_4 \in \mathbb{R}$, *let*

$$J(w_1,w_2,w_3,w_4) = \int_a^b x^{w_1}(x-\beta)^{w_2}((x-\beta)^3+(\beta-a)^3)^{w_3} \times \left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{w_4} dx, \tag{2.6}$$

then,

$$J(w_1,w_2,w_3,w_4) = \sum_{i=0}^{\infty}\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\Phi_{i,k}(t)\left[\frac{b^{w_1+k+1}-a^{w_1+k+1}}{w_1+k+1}\right], \tag{2.7}$$

where $\Phi_{i,k}(t) = \binom{w_4}{i}\binom{w_3+\theta\lambda i}{t}\binom{w_2+3t}{k}(-1)^{i+w_2+3t-k}\alpha^{\theta\lambda i}3^{-\theta\lambda i}\beta^{w_2+3t-k}(\beta-a)^{3(w_3+\theta\lambda i-t)}$ and $x \in (a, 2\beta-a)\cap(-\beta,\beta)$.

Proof. First, consider

$$J(w_1,w_2,w_3,w_4) = \int_a^b x^{w_1}(x-\beta)^{w_2}((x-\beta)^3+(\beta-a)^3)^{w_3} \times \left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{w_4} dx.$$

Applying the generalized binomial expansion of

$$\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{w_4} = \sum_{i=0}^{\infty}\binom{w_4}{i}(-1)^i\left(\frac{\alpha}{3}\right)^{\theta\lambda i}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda i}dx$$

we obtain

$$J(w_1,w_2,w_3,w_4) = \sum_{i=0}^{\infty}\binom{w_4}{i}(-1)^i\alpha^{\theta\lambda i}3^{-\alpha\lambda i}\int_a^b x^{w_1}(x-\beta)^{w_2}((x-\beta)^3+(\beta-a)^3)^{w_3-\theta\lambda i}dx.$$

Similarly, by the expansion of

$$((\beta-a)^3+(x-\beta)^3)^{w_3-\theta\lambda i} = (\beta-a)^{3(w_3-\theta\lambda i)}\sum_{t=0}^{\infty}\binom{w_3+\theta\lambda i}{t}\left(\frac{x-\beta}{\beta-a}\right)^{3t}$$

for $|\frac{x-\beta}{\beta-a}|<1$, we have

$$J(w_1,w_2,w_3,w_4) = \sum_{i=0}^{\infty}\sum_{t=0}^{\infty}\binom{w_4}{i}\binom{w_3+\theta\lambda i}{t}(-1)^i\alpha^{\theta\lambda i}3^{-\alpha\lambda i}(\beta-a)^{3(w_3-\theta\lambda i)} \times \int_a^b \frac{x^{w_1}}{(\beta-a)^{3t}}(x-\beta)^{w_2+3t}dx.$$

Expanding

$$(x-\beta)^{w_2+3t} = (-\beta)^{w_2+3t}\left(1+\left(\frac{x}{-\beta}\right)\right)^{w_2+3t}$$
$$= \sum_{k=0}^{\infty}\binom{w_2+3t}{k}(-\beta)^{w_2+3t-k}x^k$$

for $|\frac{x}{-\beta}| < 1$, we obtain

$$J(w_1,w_2,w_3,w_4) = \sum_{i=0}^{\infty}\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\binom{w_4}{i}\binom{w_3+\theta\lambda i}{t}\binom{w_2+3t}{k}(-1)^{i+w_2+3t-k}\alpha^{\theta\lambda i}3^{-\theta\lambda i}\beta^{w_2+3t-k}$$
$$\times(\beta-a)^{3(w_3+\theta\lambda i-t)}\int_a^b x^{w_1+k}dx.$$

Finally, by integration, we arrive at

$$J(w_1,w_2,w_3,w_4) = \sum_{i=0}^{\infty}\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\Phi_{i,k}(t)\left[\frac{b^{w_1+k+1}-a^{w_1+k+1}}{w_1+k+1}\right],$$

where $\Phi_{i,k}(t) = \binom{w_4}{i}\binom{w_3+\theta\lambda i}{t}\binom{w_2+3t}{k}(-1)^{i+w_2+3t-k}\alpha^{\theta\lambda i}3^{-\theta\lambda i}\beta^{w_2+3t-k}(\beta-a)^{3(w_3+\theta\lambda i-t)}$. □

Table 1 provided some numerical values of $J(w_1,w_2,w_3,w_4)$ with absolute error in parenthesis for some values of parameters and $w_1,w_2,w_3,\ w_4$. The values are obtained numerically using `R-software`. It is shown that the absolute error is small, thus we can compute other measures with reliable accuracy.

Table 1. Numerical values of $J(w1,w2,w3,w4)$ for some values of parameters.

a	b	θ	λ	δ	$w1$	$w2$	$w3$	$w4$	$J(w1,w2,w3,w4)$
0.0	5.0	1.0	4.0	5.0	7.0	0.2	0.5	0.5	1.9735 (2.2e^{-5})
0.0	5.0	1.0	0.4	5.0	0.7	0.2	0.5	0.5	1.1965 (9.0e^{-5})
0.0	5.0	1.5	0.4	5.0	0.7	1.2	1.5	2.5	0.2157 (3.7e^{-8})
0.0	5.0	1.5	0.2	0.5	1.7	1.2	1.5	2.5	0.0623 (6.7e^{-7})
0.0	5.0	1.4	0.2	1.2	1.0	2.0	1.5	2.0	0.1114 (3.6e^{-7})
0.0	5.0	0.1	0.3	1.8	1.3	4.0	1.5	3.1	0.0815 (3.8e^{-7})
0.0	5.0	3.1	2.3	3.8	9.3	2.1	7.5	6.1	1.0993 (1.1e^{-6})
-2.0	3.0	3.0	0.3	0.8	0.3	0.1	0.5	0.1	1.6828 (1.9e^{-14})
-5.0	5.0	3.0	0.3	1.1	0.3	2.0	2.5	1.1	0.3105 (3.4e^{-15})
-10	15	1.9	5.4	2.0	0.07	0.2	1.5	0.5	1.9765 (2.2e^{-14})

By considering (2.1), we can obtain the r^{th} moment and moment generating function of the KwEUq distribution which are very useful in the study of characteristics and features of a distribution. The r^{th} moment of the KwEUq distribution can be obtained using

$$E[X^r] = \int_a^b x^r f(x)dx = \delta\lambda\theta\alpha^{\theta\lambda}3^{1-\theta\lambda}\int_a^b x^r(x-\beta)^2((x-\beta)^3+(\beta-a)^3)^{\theta\lambda-1}$$
$$\times\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{\delta-1}dx,$$

hence,

$$E[X^r] = \frac{\delta\lambda\theta\alpha^{\theta\lambda}}{3^{\theta\lambda-1}} J(r, 2, \theta\lambda - 1, \delta - 1). \tag{2.8}$$

The moment generating function (mgf) of the KwEUq distribution can be computed directly using $M_X(t) = E[e^{tX}] = \sum_{r=0}^{\infty} \frac{t^r}{r!} E[X^r]$, thus, by substituting (2.8), we have

$$M_X(t) = \sum_{r=0}^{\infty} \frac{\delta\lambda\theta\alpha^{\theta\lambda} t^r}{3^{\theta\lambda-1} r!} J(r, 2, \theta\lambda - 1, \delta - 1).$$

Corollary 2.2. *Let $X \sim KwEUq$, then the variance (σ^2) , coefficient of variation (CV), skewness (γ^3) and kurtosis (γ^4) could be obtain from*

$$\sigma^2 = \mu_2^{'} - \mu_1^{'2}, \quad CV = \sqrt{\frac{\mu_2^{'}}{\mu_1^{'2}} - 1}, \quad \gamma^3 = \frac{\mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + \mu_1^{'3}}{(\mu_2^{'} - \mu_1^{'2})^{3/2}},$$

$$and \quad \gamma^4 = \frac{\mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}\mu_1^{'} + 3\mu_1^{'4}}{(\mu_2^{'} - \mu_1^{'2})^2}.$$

It is shown from the table 2 that for $a = 0$, $b = 5$, and θ, λ, δ are decreasing (i) the first six moments and kurtosis are decreasing –increasing–decreasing $\cdots$ (ii) the variance σ^2, CV, and skewness are increasing– decreasing– increasing $\cdots$

Table 2. First six moments, variance (σ^2), coefficient of variation (CV), skewness γ^3, and kurtosis γ^4 of $KwEUq$ for some parameter values and $a = 0$, $b = 5.0$.

θ	λ	δ	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	σ^2	CV	γ^3	γ^4
0.1	0.1	0.1	3.7925	18.2964	89.6003	440.678	2172.340	10725.96	3.9139	0.5216	-1.2235	7.7291
0.15	0.16	0.14	3.6192	17.2358	83.7699	409.522	2008.19	9869.14	4.1373	0.5620	-1.0168	5.0123
0.2	0.2	0.2	3.2937	15.3845	73.9514	358.364	1743.962	8512.16	4.5359	0.6466	-0.6833	2.0012
0.4	0.35	0.5	2.3244	9.9411	45.5408	212.469	1000.096	4736.150	4.5384	0.9165	0.1382	0.7410
0.7	0.5	0.8	1.9182	7.6201	33.5861	152.071	697.054	3221.280	3.9408	1.0349	0.4923	1.4610
1.0	1.5	1.8	2.3154	8.5789	35.6902	153.764	673.034	2976.110	3.2180	0.7748	0.1603	2.7637
2.0	2.5	3.5	2.4048	8.4282	33.3092	137.093	574.643	2435.99	2.6467	0.6766	0.0734	5.5101
2.6	3.5	5.8	2.7853	10.0619	39.7472	162.328	673.681	2824.584	2.3039	0.5449	-0.3181	12.050
4.0	5.0	8.0	3.4457	13.4616	55.0097	229.155	964.900	4093.794	1.5882	0.3657	-1.1596	68.272
8.0	5.0	9.0	3.3475	12.8632	51.9585	214.342	894.438	3762.204	1.6573	0.3846	-1.0296	55.956

2.2 Entropy

An entropy of a random variable X can be defined as a measure of variation of uncertainty. In this section, we consider the two most important and popular entropies known as the Shannon and Renyi entropies. The Shannon entropy measure of a random variable X with KwEUq distribution can be defined by $E[-\log f(x)]$, then, the Shannon entropy of X can be computed by the following lemma.

Lemma 2.3. *Let X be a random variable with KwEUq distribution and pdf given by (2.2), then,*

$$E[\log(x - \beta)] = \frac{\delta\lambda\theta\alpha^{\theta\lambda}}{3^{\theta\lambda-1}} \frac{\partial}{\partial t} J(0, 2 + t, \theta\lambda - 1, \delta - 1)|_{t=0}, \tag{2.9}$$

$$E[\log((x - \beta)^3 + (\beta - a)^3)] = \frac{\delta\lambda\theta\alpha^{\theta\lambda}}{3^{\theta\lambda-1}} \frac{\partial}{\partial t} J(0, 2, \theta\lambda + t - 1, \delta - 1)|_{t=0}, \tag{2.10}$$

$$E\left[\log\left(1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x - \beta)^3 + (\beta - a)^3)^{\theta\lambda}\right)\right] = \frac{\delta\lambda\theta\alpha^{\theta\lambda}}{3^{\theta\lambda-1}} \frac{\partial}{\partial t} J(0, 2, \theta\lambda - 1, \delta + t - 1)|_{t=0}. \tag{2.11}$$

Proof. We consider the first part

$$\begin{aligned}E[\log(x-\beta)] &= \int_a^b \log(x-\beta) f(x) dx \\ &= \frac{\partial}{\partial t}\int_a^b (x-\beta)^t f(x) dx|_{t=0} \\ &= \delta\lambda\theta\alpha^{\theta\lambda} 3^{1-\theta\lambda} \frac{\partial}{\partial t}\int_a^b (x-\beta)^{2+t}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda-1} \\ &\times\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{\delta-1} dx|_{t=0},\end{aligned}$$

hence,

$$E[\log(x-\beta)] = \frac{\delta\lambda\theta\alpha^{\theta\lambda}}{3^{\theta\lambda-1}}\frac{\partial}{\partial t}J(0,2+t,\theta\lambda-1,\delta-1)|_{t=0},$$

where $J(.,.,.,.,.,.)$ is given by (2.7), (2.10) and (2.11) follows similar. □

Proposition 2.2. *Let $X \sim KwEUq(\theta,\lambda,\delta,a,b)$, then, the Shannon entropy of X can be expressed as*

$$\begin{aligned}E[-\log f(x)] = &-\log\left(\frac{\delta\lambda\theta\alpha^{\theta\lambda}}{3^{\theta\lambda-1}}\right) - \frac{2\delta\lambda\theta\alpha^{\theta\lambda}}{3^{\theta\lambda-1}}\frac{\partial}{\partial t}J(0,2+t,\theta\lambda-1,\delta-1)|_{t=0} \\ &-\frac{\delta\lambda\theta\alpha^{\theta\lambda}(\theta\lambda-1)}{3^{\theta\lambda-1}}\frac{\partial}{\partial t}J(0,2,\theta\lambda+t-1,\delta-1)|_{t=0} \\ &-\frac{\delta\lambda\theta\alpha^{\theta\lambda}(\delta-1)}{3^{\theta\lambda-1}}\frac{\partial}{\partial t}J(0,2,\theta\lambda-1,\delta+t-1)|_{t=0}.\end{aligned}$$

Proof.

$$\begin{aligned}E[-\log f(x)] = &-E\left[\log(\delta\lambda\theta\alpha^{\theta\lambda}3^{1-\theta\lambda})\right] - 2E[\log(x-\beta)] \\ &-(\theta\lambda-1)E[\log((x-\beta)^3+(\beta-a)^3)] \\ &-(\delta-1)E\left[\log\left(1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right)\right],\end{aligned}$$

and in view of lemma 2.3, we obtain the result. □

The Renyi entropy of a random variable X is defined by $I_{R(\rho)} = \frac{1}{1-\rho}\log\left[\int_a^b f(x)^\rho dx\right]$, where $\rho > 0$ and $\rho \neq 1$. The Renyi entropy of X with KwEUq distribution can be obtained as follows.

$$\begin{aligned}\int_a^b f^\rho(x)dx = &\int_a^b \delta^\rho\lambda^\rho\theta^\rho\alpha^{\rho\theta\lambda}3^{\rho(1-\theta\lambda)}(x-\beta)^{2\rho}((x-\beta)^3+(\beta-a)^3)^{\rho(\theta\lambda-1)} \\ &\times\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{\rho(\delta-1)} dx.\end{aligned}$$

Applying lemma 2.1, the integral becomes

$$\int_a^b f^\rho(x)dx = \frac{\delta^\rho\lambda^\rho\theta^\rho\alpha^{\rho\theta\lambda}}{3^{\rho(\theta\lambda-1)}}J(0,2\rho,\rho(\theta\lambda-1),\rho(\delta-1)).$$

Hence, the Renyi entropy of X is

$$I_{R(\rho)} = \frac{1}{1-\rho}\log\left[\frac{\delta^\rho\lambda^\rho\theta^\rho\alpha^{\rho\theta\lambda}}{3^{\rho(\theta\lambda-1)}}J(0,2\rho,\rho(\theta\lambda-1),\rho(\delta-1))\right].$$

2.3 Order Statistics

Order statistics play an crucial role in reliability and quality control. It is also an important tool in non parametric statistic. Let $X_1, X_2, X_3, \ldots, X_n$ be a simple random sample obtained from KwEUq distribution and let $X_{1:n} \leq \cdots \leq X_{n:n}$, be the corresponding order statistics. Then for $j = 1, 2, 3, \ldots, n$, the corresponding pdf of $X_{j:n}$, say $f_{X_{j:n}}(x)$, is obtained as follows. The pdf of the j^{th} order statistics can be computed from

$$f_{X_{j:n}}(x;\alpha,\beta,\lambda,\theta,\delta) = \frac{n!}{(j-1)!(n-j)!} f(x)(F(x))^{j-1}(1-F(x))^{n-j},$$

where $f(x)$ and $F(x)$ are the pdf and cdf of KwEUq distribution. we have

$$f_{X_{j:n}}(x;\alpha,\beta,\lambda,\theta,\delta) = \sum_{m=0}^{n-j} \frac{n!(-1)^m}{(j-1)(n-j-m)!m!} f(x)(F(x))^{j+m-1}. \tag{2.12}$$

Binomial expansion of $(F(x))^{j+m-1}$ is

$$(F(x))^{j+m-1} = \sum_{l=0}^{j+m-1} \binom{j+m-1}{l}(-1)^l \left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{\delta l}. \tag{2.13}$$

Substituting (2.13) into (2.12), we have

$$f_{X_{j:n}}(x;\alpha,\beta,\lambda,\theta,\delta) = \sum_{m=0}^{n-j}\sum_{l=0}^{j+m-1} \frac{\binom{j+m-1}{l}(-1)^{m+l}n!\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{\delta l}}{(j-1)!(n-j-m)!m!} f(x)$$

Upon some algebraic manipulations, we obtain

$$f_{X_{j:n}}(x;\alpha,\beta,\lambda,\theta,\delta) = \sum_{m-0}^{n-j}\sum_{l=0}^{j+m-1} \frac{\binom{j+m-1}{l}n!(-1)^{m+l} f(x;\alpha,\beta,\lambda,\theta,\delta(l+1))}{(l+1)(j-1)!(n-j-m)!m!},$$

where $f(x;\alpha,\beta,\lambda,\theta,\delta(l+1))$ is the pdf of the KwEUq distribution with power $\delta(l+1)$. The moments of order statistics of the KwEUq distribution can be obtained using (2.3).

2.4 Probability Weighted Moments

The PWM of the random variable X with the KwEUq distribution, say $\rho_{r,s}$, is defined by

$$\rho_{r,s} = E[X^r F(X)^s] = \int_a^b x^r F^s(x) f(x)dx, \tag{2.14}$$

where $F(x)$ and $f(x)$ are the cdf and pdf of the KwEUq distribution. By expansion of $F^s(x)$ as

$$F^s(x) = \sum_{u=0}^{s} \binom{s}{u}(-1)^u \left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{\delta u}, \tag{2.15}$$

and inserting (2.15) into (2.14), we have

$$\rho_{r,s} = \sum_{u=0}^{s} \binom{s}{u}(-1)^u \int_a^b x^r \left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]^{\delta u} f(x)dx. \tag{2.16}$$

Upon some algebraic manipulations, we obtain $\rho_{r,s}$ as

$$\rho_{r,s} = \sum_{u=0}^{s} \binom{s}{u} \frac{(-1)^u}{(u+1)} \int_a^b x^r f(x;\alpha,\beta,\lambda,\theta,\delta(u+1))dx,$$

hence,

$$\rho_{r,s} = \sum_{u=0}^{s} \binom{s}{u} \frac{(-1)^u}{(u+1)} \frac{\delta(u+1)\lambda\theta\alpha^{\theta\lambda}}{3^{\theta\lambda-1}} J(r,2,\theta\lambda-1,\delta(u+1)-1).$$

3 ESTIMATION

The maximum likelihood estimation (MLE) method is one of the most popular and commonly used estimation methods for estimating the unknown parameters of a distribution. However, we found that there is an irregularity in the log-likelihood function of the KwEUq at the minimum and maximum order statistics, thus, the usual maximum likelihood method fails to exist. Thus, the use of alternative maximum likelihood estimation (AMLE) and modified maximum likelihood estimation (MMLE) are employed.

3.1 Maximum Likelihood Estimation

Let $X_1, X_2, X_3, \ldots, X_n$ be an independent and identically distributed random sample from KwEUq distribution and let $X_{1:n} \leq \cdots \leq X_{n:n}$, be the order statistics from this sample. Let $\Theta = (\delta, \theta, \lambda)^T$ and let the approximate maximum likelihood estimates be $\hat{\Theta}_{MLE} = (\hat{\delta}, \hat{\theta}, \hat{\lambda})$, then, the the likelihood function $l(\Theta)$ of KwEUq distribution for complete data set is given by

$$\begin{aligned}\log l(\Theta) &= n\log\delta + n\log\lambda + n\log\theta + \theta\lambda n\log\alpha + (1-\theta\lambda)n\log 3 \\ &+ 2\sum_{i=1}^{n}\log(x_i-\beta) + (\theta\lambda-1)\sum_{i=1}^{n}\log((x_i-\beta)^3+(\beta-a)^3) \\ &+ (\delta-1)\sum_{i=1}^{n}\log\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right].\end{aligned} \tag{3.1}$$

The first partial derivative of $\log l(\Theta)$ with respect to δ, θ and λ are respectively given by

$$\frac{\partial(\log l(\Theta))}{\partial\delta} = \frac{n}{\delta} + \sum_{i=1}^{n}\log\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right] \tag{3.2}$$

$$\begin{aligned}\frac{\partial(\log l(\Theta))}{\partial\theta} &= \frac{n}{\theta} + n\log\alpha - \lambda n\log 3 + \lambda\sum_{i=1}^{n}\log((x_i-\beta)^3+(\beta-a)^3) \\ &+ (\delta-1)\sum_{i=1}^{n}\frac{(-1)(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3))^{\theta\lambda}\ln\left(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3)\right)^{\lambda}}{\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]}\end{aligned} \tag{3.3}$$

$$\frac{\partial(\log l(\Theta))}{\partial\lambda} = \frac{n}{\lambda} - \theta n\log 3 + (\theta-1)\sum_{i=1}^{n}\log((x_i-\beta)^3+(\beta-a)^3)$$
$$+(\delta-1)\sum_{i=1}^{n}\frac{\left(-1\right)(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3))^{\theta\lambda}\ln\left(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3)\right)^{\theta}}{\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]}. \quad (3.4)$$

We can determined $\hat{\delta}$ from (3.2) if the value of $\hat{a}, \hat{b}, \hat{\lambda}$ and $\hat{\theta}$ are known,

$$\delta = \frac{-n}{\sum_{i=1}^{n}\log\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]}. \quad (3.5)$$

This can be computed by solving (3.2), (3.3) and (3.4), but the following proposition describes the irregularity of the MLE method.

Proposition 3.1. *Let $\ell(\theta,\lambda,\delta,a,b|\boldsymbol{x})$ denote the likelihood function of an independent and identically distributed (i.i.d) random sample of size $n \geq 1$ say $x_1, x_2, \cdots, x_n$, drawn from KwEUq$(\theta,\lambda,\delta,a,b)$ distribution and let $a = X_1 \leq X_2, \leq \cdots \leq X_n = b$ be the order statistics from the sample, then, for $x_i = x_j$ or for $x_i \neq x_j$, the $\log\ell(\Theta)$ diverges at x_1 or x_n.*

Proof. Consider (3.1), at x_1, $\log((x_i-\beta)^3+(\beta-a)^3) = \log(0) = -\infty$, while at x_n, $\log\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right] = \log(0) = -\infty$. □

3.1.1 Alternative Maximum Likelihood Estimation (AMLE)

In the alternative maximum likelihood, firstly we set $a = X_1$ and $b = X_n$. Then we delete all the data points corresponding to the minimum and maximum order statistics. Then applying the usual maximum likelihood technique to obtain the approximate alternative maximum likelihood estimates, say $\hat{\Theta}_{AMLE} = (\hat{\theta}, \hat{\delta}, \hat{\lambda})$ numerically by simultaneously solving (3.6),(3.7) and (3.8) using some mathematical package such as R-software.

$$\frac{\partial(\log l(\Theta))}{\partial\delta} = \frac{n}{\delta} + \sum_{x_i\neq x_1,x_n}^{n}\log\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right] = 0 \quad (3.6)$$

$$\frac{\partial(\log l(\Theta))}{\partial\theta} = \frac{n}{\theta} + n\log\alpha - \lambda n\log 3 + \lambda\sum_{x_i\neq x_1,x_n}^{n}\log((x_i-\beta)^3+(\beta-a)^3)$$
$$+(\delta-1)\sum_{x_i\neq x_1,x_n}^{n}\frac{\left(-1\right)(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3))^{\theta\lambda}\ln\left(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3)\right)^{\lambda}}{\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]} = 0 \quad (3.7)$$

$$\frac{\partial(\log l(\Theta))}{\partial\lambda} = \frac{n}{\lambda} - \theta n\log 3 + (\theta-1)\sum_{x_i\neq x_1,x_n}^{n}\log((x_i-\beta)^3+(\beta-a)^3)$$
$$+(\delta-1)\sum_{x_i\neq x_1,x_n}^{n}\frac{\left(-1\right)(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3))^{\theta\lambda}\ln\left(\left(\frac{\alpha}{3}\right)((x_i-\beta)^3+(\beta-a)^3)\right)^{\theta}}{\left[1-\left(\frac{\alpha}{3}\right)^{\theta\lambda}((x_i-\beta)^3+(\beta-a)^3)^{\theta\lambda}\right]} = 0 \quad (3.8)$$

3.1.2 Modified Maximum Likelihood Estimation (MMLE)

For the modified Maximum Likelihood, set $a = X_1$ and $b = X_n$ then, we shift the data points corresponding to the minimum and maximum order statistics by introducing appropriate τ, $\pi \in \mathbb{R} - \{0\}$ such that the new values correspond to x_1 and x_n say $x_1^* = x_1 + \tau$ and $x_n^* = x_n + \pi$. The approximate modified maximum likelihood estimates, say $\hat{\Theta}_{MMLE} = (\hat{\delta}, \hat{\theta}, \hat{\lambda})$ are obtained setting (3.9), (3.10) and (3.11) equal to zero and solving them using some mathematical packages such as `nlm, optimx in R-software`.

$$\frac{\partial(\log l(\Theta))}{\partial\delta} = \frac{n}{\delta} + \sum_{i=1}^{n} \log\left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x_i - \beta)^3 + (\beta - a)^3)^{\theta\lambda}\right]|_{x_1 = x_1^*, x_n = x_n^*}, \tag{3.9}$$

$$\frac{\partial(\log l(\Theta))}{\partial\theta} = \frac{n}{\theta} + n\log\alpha - \lambda n \log 3 + \lambda \sum_{i=1}^{n} \log((x_i - \beta)^3 + (\beta - a)^3)$$
$$+ (\delta - 1)\sum_{i=1}^{n} \frac{\left(-1)(\left(\frac{\alpha}{3}\right)((x_i - \beta)^3 + (\beta - a)^3)\right)^{\theta\lambda} \ln\left(\left(\frac{\alpha}{3}\right)((x_i - \beta)^3 + (\beta - a)^3)\right)^{\lambda}}{\left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x_i - \beta)^3 + (\beta - a)^3)^{\theta\lambda}\right]}|_{x_1 = x_1^*, x_n = x_n^*}, \tag{3.10}$$

$$\frac{\partial(\log l(\Theta))}{\partial\lambda} = \frac{n}{\lambda} - \theta n \log 3 + (\theta - 1)\sum_{i=1}^{n} \log((x_i - \beta)^3 + (\beta - a)^3)$$
$$+ (\delta - 1)\sum_{i=1}^{n} \frac{\left(-1)(\left(\frac{\alpha}{3}\right)((x_i - \beta)^3 + (\beta - a)^3)\right)^{\theta\lambda} \ln\left(\left(\frac{\alpha}{3}\right)((x_i - \beta)^3 + (\beta - a)^3)\right)^{\theta}}{\left[1 - \left(\frac{\alpha}{3}\right)^{\theta\lambda} ((x_i - \beta)^3 + (\beta - a)^3)^{\theta\lambda}\right]}|_{x_1 = x_1^*, x_n = x_n^*}. \tag{3.11}$$

3.2 Simulation

Simulations have been carried out to investigate the estimators of the parameters of the KwEUq distribution. We generated 10,000 samples of size $n = 20, 30$, 50, 100 and 150 from the KwEUq distribution for some values of λ, δ, θ, and $a = 0$, $b = 5$. Let $\Theta = (\delta, \theta, \lambda)^T$ be the vector of the unknown parameters of KwEUq distribution. The unknown parameters were determined by the alternative maximum likelihood and modified maximum likelihood methods as follows

(i) The alternative maximum likelihood estimates are obtained by equating the nonlinear equations (3.6), (3.7) and (3.8) to zero and solve them. Moreover, we set $a = X_1$ and $b = X_n$ and delete all the data points corresponding to X_1 and X_n. We denote by $\hat{\Theta}_A = (\hat{\delta}_A, \hat{\theta}_A, \hat{\lambda}_A)^T$ the vector of the estimators of Θ by AMLE.

(ii) The modified maximum likelihood estimates were obtained by equating the nonlinear equations (3.9), (3.10) and (3.11) to zero and solve them. In this case, $a = X_1$, $b = X_n$ and all the data points corresponding to X_1 are replaced by $X_1^* = X_1 + \tau$, we choose $\tau = (\frac{X_2 - X_1}{2})$, also, all the data points corresponding to X_n are replaced by $X_n^* = X_n - \pi$, we choose $\pi = \frac{X_n - X_{n-1}}{2}$. We denote by $\check{\Theta}_M = (\check{\delta}_M, \check{\theta}_M, \check{\lambda}_M)^T$ the vector of the estimators of Θ by MMLE.

The validity of the two methods i.e AMLE and MMLE are discussed by the average bias and average mean square error of the simulated estimates.

(i) The average bias of the simulated $10,000$ estimates of Θ is obtained by $Bias(\hat{\Theta}) = \frac{1}{10000}\sum_{i=1}^{10000}(\hat{\Theta}_i - \Theta)$.

(ii) The average mean square error of the simulated $10,000$ estimates of Θ is given by $MSE(\hat{\Theta}) = \frac{1}{10000}\sum_{i=1}^{10000}(\hat{\Theta}_i - \Theta)^2$.

Clearly from table 3, the two methods perform consistently and (i) as the sample size increases the AMLEs and MMLEs converges to their true value (ii) in both methods the MSE decreases as the sample size increases (iii) the average bias is also decreasing as the sample size increases in some cases and also the average bias is negative in some cases.

Table 3. AMLEs, MMLEs, Bias and MSE for the simulated data set with $a = 0, b = 5$

λ	δ	θ	n	$\hat{\lambda}_A$	$\check{\lambda}_M$	$\hat{\delta}_A$	$\check{\delta}_M$	$\hat{\theta}_A$	$\check{\theta}_M$	$B(\hat{\lambda}_A)$	$B(\check{\lambda}_M)$	$B(\hat{\delta}_A)$	$B(\check{\delta}_M)$	$B(\hat{\theta}_A)$	$B(\check{\theta}_M)$	MSE($\hat{\lambda}_A$)	$MSE(\check{\lambda}_M)$	$MSE(\hat{\delta}_A)$	$MSE(\check{\delta}_M)$	$MSE(\hat{\theta}_A)$	$MSE(\check{\theta}_M)$
0.2	0.4	0.3	20	0.3118	0.2637	0.5945	0.4925	0.3711	0.3380	0.1118	0.0637	0.1946	0.0925	0.0711	0.0380	0.0182	0.0087	0.0828	0.0334	0.0091	0.00420
			30	0.2909	0.2468	0.5339	0.4595	0.3516	0.3241	0.0909	0.0468	0.1339	0.0595	0.0516	0.0241	0.0114	0.0047	0.0366	0.0150	0.0044	0.00180
			50	0.2768	0.2356	0.4919	0.4386	0.3405	0.3161	0.0768	0.0356	0.0919	0.03856	0.0405	0.0161	0.0075	0.0026	0.0166	0.0069	0.0025	0.00085
			100	0.2683	0.2242	0.4625	0.4236	0.3340	0.3113	0.0683	0.0242	0.0625	0.0236	0.0340	0.0113	0.0055	0.0013	0.0073	0.0027	0.0016	0.00036
			150	0.2655	0.2197	0.4526	0.4183	0.3331	0.3097	0.0655	0.0197	0.0526	0.0183	0.0331	0.0097	0.0048	0.0009	0.0049	0.0016	0.0014	0.00030
0.5	0.5	0.5	20	0.6307	0.5553	0.7118	0.5752	0.6307	0.5553	0.1307	0.0553	0.2118	0.0752	0.1307	0.0553	0.0319	0.0141	0.1090	0.0383	0.0319	0.0141
			30	0.5923	0.5348	0.6399	0.5443	0.5923	0.5348	0.0923	0.0348	0.1399	0.0443	0.0923	0.0348	0.0168	0.0078	0.0468	0.0184	0.0168	0.0078
			50	0.5601	0.5195	0.5870	0.5241	0.5601	0.5195	0.0601	0.0195	0.0870	0.0241	0.0601	0.0195	0.0079	0.0040	0.0193	0.0090	0.0079	0.0040
			100	0.5352	0.5101	0.5488	0.5119	0.5352	0.5101	0.0352	0.0101	0.0488	0.0119	0.0352	0.0101	0.0032	0.0019	0.0071	0.0040	0.0032	0.0019
			150	0.5256	0.5068	0.5355	0.5084	0.5256	0.5068	0.0256	0.0068	0.0355	0.0084	0.0256	0.0068	0.0019	0.0012	0.0041	0.0025	0.0019	0.0012
0.8	0.7	0.9	20	0.9266	0.7898	0.9241	0.7123	1.0102	0.8920	0.1266	-0.0102	0.2241	0.0123	0.1102	-0.0080	0.0484	0.0236	0.1742	0.0508	0.0378	0.0183
			30	0.8870	0.7808	0.8425	0.6916	0.9764	0.8837	0.0870	-0.0192	0.1425	-0.0084	0.0764	-0.0164	0.0258	0.0147	0.0685	0.0268	0.0200	0.0113
			50	0.8608	0.7839	0.7968	0.6915	0.9527	0.8857	0.0608	-0.0161	0.0968	-0.0085	0.0527	-0.0143	0.0137	0.0087	0.0329	0.0165	0.0104	0.0067
			100	0.8378	0.7903	0.7541	0.6915	0.9318	0.8906	0.0378	-0.0097	0.0541	-0.00851	0.0318	-0.0094	0.0061	0.0044	0.0125	0.0076	0.0046	0.0034
			150	0.8272	0.7921	0.7377	0.6919	0.9227	0.8918	0.0272	-0.0079	0.0377	-0.0081	0.0227	-0.0082	0.0037	0.0028	0.0071	0.0048	0.0029	0.0022
1.0	1.5	1.2	20	1.0079	0.8496	1.5656	1.1027	1.1821	1.0553	0.0079	-0.1504	0.0656	-0.3974	-0.0179	-0.1447	0.0399	0.0466	0.5111	0.3186	0.0207	0.0397
			30	0.9860	0.8640	1.4677	1.1195	1.1730	1.0668	-01400	-0.1360	-0.0323	-0.3805	-0.0270	-0.1333	0.0212	0.0340	0.2273	0.2460	0.0142	0.0312
			50	0.9785	0.8917	1.4289	1.1741	1.1721	1.0893	-0.0215	-0.1083	-0.0711	-0.3259	-0.0279	-0.1107	0.0115	0.0212	0.1139	0.1698	0.0094	0.0212
			100	0.9780	0.9244	1.4185	1.2529	1.1752	1.1205	-0.0220	-0.0756	-0.0815	-0.2471	-0.0248	-0.0795	0.0056	0.0105	0.0573	0.0982	0.0052	0.1167
			150	0.9822	0.9412	1.4247	1.2972	1.1782	1.1371	-0.0178	-0.0588	-0.0753	-0.2028	-0.0218	-0.0629	0.0037	0.0068	0.0380	0.0669	0.0036	0.0073
1.5	1.9	2.5	20	1.3840	1.0625	1.9265	1.2052	2.3351	2.1302	-0.1160	-0.4375	0.0265	-0.6949	-0.1649	-0.3698	0.2982	0.3234	4.1366	0.9765	0.1179	0.2431
			30	1.2965	1.0457	1.6245	1.1426	2.3298	2.1229	-0.2035	-0.4543	-0.2755	-0.7574	-0.1702	-0.3771	0.1502	0.2855	0.7923	0.7936	0.0882	0.2288
			50	1.2627	1.0697	1.5192	1.1734	2.3207	2.1608	-0.2373	-0.4303	-0.3809	-0.7276	-0.1793	-0.3392	0.1116	0.2323	0.3909	0.6381	0.0671	0.1684
			100	1.2711	1.1459	1.5142	1.2832	2.3214	2.2159	-0.2289	-0.3541	-0.3858	-0.6168	-0.1786	-0.2841	0.0777	0.1517	0.2356	0.4391	0.0531	0.1100
			150	1.2917	1.1965	1.5425	1.3598	2.3331	2.2506	-0.2083	-0.3035	-0.3575	-0.5402	-0.1669	-0.2494	0.0605	0.1100	0.1877	0.3373	0.0438	0.0844
2.0	2.0	1.5	20	1.8105	1.5851	1.6303	1.0748	1.3056	1.0429	-0.1896	-0.4149	-0.3697	-0.9252	-0.1944	-0.4571	0.1119	0.2502	1.2117	1.0851	0.1680	0.2903
			30	1.8018	1.6040	1.5180	1.0992	1.2756	1.0707	-0.1982	-0.3961	-0.4820	-0.9008	-0.2245	-0.4293	0.0897	0.2144	0.5811	0.9374	0.1198	0.2365
			50	1.7997	1.6546	1.4998	1.1821	1.2786	1.1229	-0.2003	-0.3455	-0.5002	-0.8179	-0.2215	-0.3771	0.0722	0.1573	0.4156	0.7542	0.0860	0.1772
			100	1.8137	1.7129	1.5263	1.3092	1.2994	1.2024	-0.1863	-0.2871	-0.4737	-0.6908	-0.2006	-0.2976	0.0515	0.1047	0.3035	0.5318	0.0587	0.1073
			150	1.8313	1.7561	1.5771	1.4027	1.3268	1.2519	-0.1687	-0.2439	-0.4230	-0.5973	-0.1732	-0.2481	0.0415	0.0761	0.2375	0.4015	0.0427	0.0753

4 APPLICATION

In this section, we provide an application of the KwEUq distribution to a real data set. The parameter estimation of the KwEUq was achieved by alternative maximum likelihood and modified maximum likelihood estimation method. Moreover, for the MMLE we set $X_1^* = X_1 + \tau$, and $\tau = (\frac{X_2 - X_1}{2})$, also, $X_n^* = X_n - \pi$, and $\pi = \frac{X_n - X_{n-1}}{2}$. We used the Kolmogorov - Smirnov test (K-S) to compare the KwEUq and some other existing distributions. The model with the smallest value of K-S fit the data better than the other distributions. The competing distributions are:

- Poisson odd-exponential uniform (POE-U) [31] with cdf $F(x) = \frac{1-e^{-\lambda\left(1-e^{-\theta(\frac{x}{b-x})}\right)}}{1-e^{-\lambda}}, \ x \in [0, b]$
- Half logistic Poisson (HLP) [32] with cdf $F(x) = \frac{1-e^{-\lambda\left(\frac{1-e^{-ax}}{1+e^{-ax}}\right)}}{1-e^{-\lambda}}, \ x > 0$
- Exponentiated U-quadratic (EUq) [30] with cdf given in (1.3)
- Transmuted U-quadratic distribution (TUq) [33] with cdf $F(x) = (1+\lambda)\left(\frac{\alpha}{3}((x-\beta)^3 + (\beta-a)^3)\right) - \lambda\left(\frac{\alpha}{3}((x-\beta)^3 + (\beta-a)^3)\right)^2, \ x \in [a, b]$
- Exponentiated generalized U-quadratic distribution (EGUq) [34] with cdf $F(x) = (1-(1-\frac{\alpha}{3}((x-\beta)^3 + (\beta-a)^3))^\lambda)^\delta, \ x \in [a, b]$
- Mustapha type-I distribution (Mu-I) [35] with cdf $F(x) = \frac{(a+1)(x/b)}{(a+(x/b))}, \ x \in [0, b]$
- Kumaraswamy Exponential distribution (KwE) [36] with cdf $F(x) = 1 - [1-(1-e^{-ax})^\lambda]^\delta, \ x > 0$

- Kumaraswamy power distribution (KwPw) [37] with cdf $F(x) = 1 - [1 - (1 - (x/b)^{\theta})^{\lambda}]^{\delta},\ x \in [0, b]$.
- Generalized BurrXII poisson (GBXIIP) [38] with cdf $F(x) = \left(\frac{1-e^{\lambda((1+x^{\delta})^{-\theta}-1)}}{1-e^{-\lambda}}\right)^{a}\ \ x > 0$

Where $a, b, \lambda, \theta,$ and $\delta > 0$.

The data set is the remission times (in months) of a random sample of 128 bladder cancer patients provided by [39] also studied by [40], the data set are: 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29 , 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17 , 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06 , 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90 , 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23 , 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63 , 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05 , 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71 , 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76 , 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51,6.54, 8.53 , 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93 , 8.65, 12.63, 22.69.

The numerical value of the estimators, $\ell(\Theta)$ and the KS are given in table 4. The result shows that KwEUq has the smallest value of KS thus, KwEUq fit the data better than the other models. It is seen that both AMLE and MMLE methods are suitable for the estimation of the parameters of KwEUq distribution. Figure 4 display the plots of the (i) histogram and the fitted KwEUq (ii) empirical and estimated cdfs obtained by the AMLE and MMLE methods for the given data set.

Table 4. Estimated values, $\ell(\Theta)$, KS and p values for the given data set

	θ	λ	δ	a	b	τ	π	$\ell(\Theta)$	KS	p-value
KwEUq$_{AMLE}$	1.5648	0.9772	7.6645	0.2	79.05	-	-	-394.85	0.0476	0.9414
KwEUq$_{MMLE}$	3.0646	0.4723	6.7660	0.2	79.05	0.1	16.465	-404.60	0.0403	0.9880
EUq	0.5687	-	-	0.2	79.05	-	-	-451.87	0.2768	$1.11e^{-8}$
TUq	-	0.9759	-	0.2	79.05	-	-	-434.45	0.2457	$6.28e^{-7}$
EGUq	-	5.1651	1.17568	0.2	79.05	-	-	-396.09	0.0523	0.8863
KwE	-	1.5003	0.3379	0.3462	-	-	-	-401.38	0.0679	0.6169
KwPw	0.4010	2.6621	8.7118	-	79.05	-	-	-393.61	0.0749	0.4894
Mu-I	0.0970	-	-	-	79.05	-	-	-413.17	0.1239	0.0445
POE-U	1.5503	6.7828	-	-	100	-	-	-415.15	0.0808	0.3742
HLP	-	0.2015	-	0.8880	-	-	-	-413.17	0.0963	0.1863
GBXIIP	2.2192	0.0552	0.6125	13.9734	-	-	-	-409.16	0.0881	0.2914

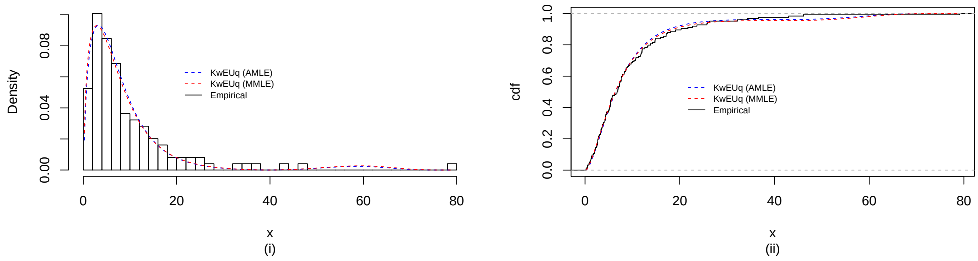

Fig. 4. Plots of the (i) histogram and the fitted KwEUq (ii) empirical cdf and estimated cdf of the KwEUq estimated by the AMLE and MMLE methods for the given data set.

5 CONCLUTIONS

The study provides a new probability model called Kumaraswamy exponentiated U-quadratic (KwEUq). Several mathematical and statistical properties were investigated such as the quantile function, moments, moment generating function, order statistics, probability weighted moments, Shannon entropy and Renyi entropy. Parameter estimation was achieved by the alternative maximum likelihood estimation (AMLE) and modified maximum likelihood estimation (MMLE) due to the irregularity found in the usual maximum likelihood estimation method. The finite sample properties of the alternative maximum-likelihood estimators (AMLEs) and modified maximum-likelihood estimators (MMLEs) are investigated by simulation studies. The simulation showed that any of the two methods can be used as the initial choice for the parameter estimation of the KwEUq. Further, we fitted the KwEUq to a real data and compare the fit with some other existing distributions as measured by Kolmogorov Smirnov test which showed that KwEUq provide better fit in compared to other models.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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