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Extending Real (C_1, C_2)-holder Valuation T0 Skew Polynomial Ring

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

The aim of this paper is as to study real (C_1, C_2)- Holder valuations on skew polynomials rings. Let D be a division ring, T be a variable over D,σ an endomorphism of D, δ a σ-derivation of D and $R = D[T;\sigma;\delta]$ the left skew polynomial ring over D. We show the set $(HVal_\nu(R), \preceq)$ of σ-compatible real Holder valuations which extend as to R a fixed proper real Holder valuation $\subseteq$ on D, has a natural structure of parameterized complete non-metric, where $\preceq$ is the partial order given by $\mu \preceq \mu^{'}$, if and only if $\mu(f) \leq \mu^{'}(f)$, for all $f \in R$ and $\mu, \mu^{'} \in HVal_\nu(R)$.

Keywords: Krull valuations; (C_1, C_2)- Holder valuations; skew polynomial ring.

1 Introduction and Preliminaries

Throughout this paper, let D be a division ring, T a variable over D, σ an endomorphism of D, δ a σ- derivation of D(i.e. for each $a, b \in R$,

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$\delta(a+b) = \delta(a) + \delta(b), \delta(ab) = \sigma(a)\delta(b) + \delta(a)b$) and
$R = D[T;\sigma;\delta] = \{f(T) = a_nT^n + \cdots + a_1T + a_0 | a_i \in D, i \in \{0,1,2,\cdots n\}\}$ left skew polynomial ring over D [1], such that $Ta = \sigma(a)T + \delta(a)$. A.Granja [2] studied real valuations on skew polynomials rings, we in paper have generalized to real Holder valuation on skew polynomial rings.

Definition 1.1. A valuation on $R = D[T, \sigma, \delta]$ is a map $\nu : R \to \bar{\mathbf{R}}$ such that
(V1) $\nu(f+g) = \nu(f) + \nu(g)$ for all $f, g \in R$;
(V2) $\nu(f+g) \geq Min\{\nu(f), \nu(g)\}$ for all $f, g \in R$;
(V3) $\nu(1) = 0$ and $\nu(1) = 0$.
where $\bar{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ is the extended monoid of $\mathbf{R}$ by a symbol ∞ satisfying the usual rules $\infty + x = x + \infty = \infty$ for all $x \in \bar{\mathbf{R}}$ and $x < \infty$ for all $x \in \mathbf{R}$. If $\mu(R) = \{o, \infty\}$, μ is said to be trivial, otherwise two-side ideal $\mu^{-1}(\infty)$ of R is called the support of μ and valuation on $R = D[T, \sigma, \delta]$ with zero support are called Krull valuations.

Definition 1.2. A (C_1, C_2)- Holder valuation on $R = D[T, \sigma, \delta]$ is a map such that $C_1 \geq 1, C_2 \geq 1$ and
(HV1) $C_1^{-1}(\mu(f) + \mu(g)) \leq \mu(fg) \leq C_1(\mu(f) + \mu(g))$ for all $f, g \in R$;
(HV2) $\mu(f+g) \geq C_2 Min\{\mu(f), \mu(g)\}$ for all $f, g \in R$;
(HV3) $\mu(0) = \infty, \mu(1) = 0$.
where $\bar{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ is the extended monoid of $\mathbf{R}$ by a symbol ∞ satisfying the usual rules $\infty + x = x + \infty = \infty$ for all $x \in \bar{\mathbf{R}}$ and $x < \infty$ for all $x \in \mathbf{R}$. If $\mu(R) = \{o, \infty\}$, μ is said to be (C_1, C_2)- Holder trivial, otherwise two-side ideal $\mu^{-1}(\infty)$ of R is called the support of μ and (C_1, C_2)- Holder valuation on $R = D[T, \sigma, \delta]$ with zero support are called Krull (C_1, C_2)- Holder valuations.

Let Hval(R) be the set of functions $\mu : R \to \bar{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ satisfying the standard axioms of Holder- valuations, whose restriction to D is no trivial and is σ-compatible (i.e. $\mu(\sigma(a)) = \mu(a)$ for each $a \in D$).

We consider the partial order $\preceq$ on $HVal(R)$ given by $\mu \preceq \mu'$ if and only if $\mu(f) \leq \mu'(f)$ for all $f \in R$ and $\mu, \mu' \in HVal(R)$.
Since $\mu \preceq \mu'$ implies that μ and μ' have the same restriction to D (see Remark 2.1 below).
Let $\mu, \mu' \in HVal(R)$ be such that $\mu \prec \mu'$ and let $\varphi \in R$ be such that
$\mu(\varphi) < \mu'(\varphi)$ and $deg(\varphi) \leq deg(\varphi')$ for all $\varphi' \in R$ with $\mu(\varphi') < \mu'(\varphi')$. Here, $deg(f)$ denotes the usual degree of $f \in R$. Since for each $\mu \in HVal(R)$ and $g, f \in R$,we have

$$C_1^{-2}\mu(fg) \leq \mu(gf) \leq C_1^2\mu(fg),$$

thus

$$I(\sigma, \delta, \mu, \mu', \varphi) = min\{\mu(r(\varphi, g)) - \mu(g); g \in R, 0 \leq deg(g) < deg(\varphi)\}$$

$$\geq \mu'(\varphi) > \mu(\varphi),$$

where

$$\varphi g = q(\varphi, g)\varphi + r(\varphi, g),$$

with $deg(r(\varphi, g)) < deg(\varphi)$, and $degq(\varphi, g) = deg(g)$. i.e the left division of φg by φ(see [3]).
(Note that $\mu(r(\varphi, g)) = \mu'(r(\varphi, g))$ and $\mu'(g) = \mu(g)$,
since $deg(r(\varphi, g)) < deg(\varphi)$ and $deg(g) < deg(\varphi)$). We call $I(\sigma, \delta, \mu, \mu', \varphi)$ the compatibility index of φ with respect to φ and φ' and we point out that $I(\sigma, \delta, \mu, \mu', \varphi) = \infty$ when $\sigma = 1_D$ is the identity on D and $\delta = 0$.

2 Ordering Holder Valuations

In this section, we review some concepts about Holder- valuations on rings and we introduce some notation.

From now we shall make the assumption that every (C_1, C_2)- Holder valuation on $R = D[T, \sigma, \delta]$ is σ-compatible

(i.e. $\mu(\sigma(a)) = \mu(a)$ for all $a \in D$) and also every real (C_1, C_2)- Holder valuation on D will be assumed σ- compatible.
Finally, we denote by $deg(f)$ the usual degree of $f \in R$ (here $deg(0) = \infty$) and we also recall that if $f, g \in R$, there exist $q, r \in R$ such that $deg(r) < deg(g)$ and $f = qg + r$, i.e. we have a left division algorithm on R (see [3]).

The rest of the section is devoted to introduce and study a natural partial order $\preceq$ on the set of real Holder valuations on R: Namely, let $\mu, \bar{\mu} : R \to \bar{\mathbf{R}}$ be two real (C_1, C_2)- Holder valuation on R. We write $\mu \preceq \bar{\mu}$ if and only if $\mu(f) \leq \bar{\mu}(f)$ for all $f \in R$:
Remark 2.1. Note that if $\mu \preceq \bar{\mu}$, then $\mu(a) = \bar{\mu}(a)$ for all $a \in D$ (i.e. μ and $\bar{\mu}$ are extensions to R of the same Krull (C_1, C_2)- Holder valuation μ on D) [4]. In particular, μ is trivial on D if and only if $\bar{\mu}$ is also trivial on D.

Lemma 2.2. If μ is (C_1, C_2)- Holder valuation on $R = D[T, \sigma, \delta]$, then for each $n \geq 2$ we have:
i)
$$(2C_1^{1-n} + C_1^{2-n} + \cdots + C_1^{-1})\mu(T) \leq \mu(T^n) \leq (2C_1^{n-1} + C_1^{n-2} + \cdots + C_1)\mu(T);$$
ii)
$$\mu(a_nT^n) \geq (2C_1^{-n} + C_1^{1-n} + \cdots + C_1^{-2})\mu(T).$$

Proof. i) By induction on n, if$n = 2$, then
$$2{C_1}^{-1}\mu(T) = {C_1}^{-1}((\mu(T) + \mu(T))$$
$$\leq \mu(T^2) \leq C_1(\mu(T) + \mu(T)) = 2C_1\mu(T).$$
Let for $n \geq 2$,we have
$$(2C_1^{1-n} + C_1^{2-n} + \cdots + C_1^{-1})\mu(T) \leq \mu(T^n)$$
$$\leq (2C_1^{n-1} + C_1^{n-2} + \cdots + C_1)\mu(T).$$
then
$$\mu(T^{n+1}) = \mu(T^nT) \leq C_1(\mu(T^n + \mu(T))$$
$$\leq C_1((2C_1^{n-1} + C_1^{n-2} + \cdots + C_1)\mu(T) + \mu(T)) =$$
$$= C_1(2C_1^{n-1} + C_1^{n-2} + \cdots + C_1 + 1)\mu(T)$$
$$= (2C_1^{n} + C_1^{n-1} + \cdots + C_1)\mu(T).$$
one sided
$$\mu(T^{n+1}) = \mu(T^nT) \geq {C_1}^{-1}(\mu(T^n) + \mu(T))$$

$$\geq {C_1}^{-1}((2C_1^{1-n} + C_1^{2-n} + \cdots + {C_1}^{-1})\mu(T) + \mu(T)) =$$

$$= {C_1}^{-1}(2C_1^{1-n} + C_1^{2-n} + \cdots + {C_1}^{-1} + 1)\mu(T) =$$

$$= (2C_1^{1-(n+1)} + C_1^{1-n} + \cdots + {C_1}^{-1})\mu(T).$$

ii)

$$\mu(a_nT^n) \geq C_1^{-1}(\mu(a_n) + \mu(T^n)) = C_1^{-1}\mu(T^n)$$

$$\geq C_1^{-1}(2C_1^{1-n} + C_1^{2-n} + \cdots + C_1^{-1})\mu(T) = (2C_1^{-n} + C_1^{1-n} + \cdots + C_1^{-2})\mu(T).$$

□

Corollary 2.3. If $\mu(T) \geq 0$, then $\mu(h) \geq 0$ for all $h \in R$.

Proof. Let $h \in R$. Then

$$h = a_nT^n + a_{n-1}T^{n-1} + \cdots + a_1T + a_0,$$

thus

$$\mu(h) \geq C_2Min\{\mu(a_nT^n), \mu(a_{n-1}T^{n-1} + \cdots + a_1T + a_0)\}$$
$$\geq C_2Min\{\mu(a_nT^n), \mu(a_{n-1}T^{n-1}), \cdot\cdot, \mu(a_1T), \mu(a_0)\}.$$

Hence by assumption and by lemma 2.2 we have $\mu(a_iT^i) \geq 0$
(for $i \in \{0, 1, 2 \cdot\cdot, n\}$). so $\mu(h) \geq 0$. □

Next, we shall describe the real (C_1, C_2)- Holder valuations μ on R whose restriction to D is trivial. We have the following possibilities:
A) There exists $h \in R$ such that $\mu(h) < 0$. Then by corollary 2.3, $\mu(T) < 0$;
B) $\mu(h) \geq 0$ for all $h \in R$.

Lemma 2.4. Let $\mu(h) \geq 0$ for all $h \in R$. Then $A_\mu = \{h \in R; \mu(h) > 0\}$ is two- side ideal of R and $A_\mu = Rf$ for some irreducible element $f \in R$.

Proof. 1) let $h, h' \in A_\mu$. Then $\mu(h) > 0$ and $\mu(h') > 0$. Hence,

$$\mu(h + h') \geq C_2Min\{\mu(h), \mu(h')\} > 0.$$

Therefore $h + h' \in A_\mu$.
2) Let $f \in R, h \in A_\mu$.Then $\mu(f) \geq 0$ and $\mu(h) > 0$. Hence,

$$\mu(hf) \geq C_1^{-1}(\mu(h) + \mu(f)) \geq C_1^{-1}\mu(h) > 0$$

and

$$\mu(fh) \geq C_1^{-1}(\mu(f) + \mu(h)) \geq C_1^{-1}\mu(h) > 0.$$

Thus $hf, fh \in A_\mu$.
Therefore A_μ is the two-side ideal of R and Since R is a left principal ideal domain (see [3]), thus $A_\mu = Rf$ for some irreducible element $f \in R$.

□

Proposition 2.5. let $\mu(h) \geq 0$ for all $h \in R$, $A_\mu = \{h \in R; \mu(h) > 0\} = Rf$. Then we have :
B1) If $A_\mu = (0)$, then μ is a trivial (C_1, C_2)- Holder valuation on R.
B2) If $A_\mu \neq (0)$, then for all $g \in R - \{0\}$, such that $\mu(g) \neq 0$, we have

$$(2C_1^{-n} + C_1^{1-n} + \cdots + C_1^{-2})\mu(f) \leq \mu(g) \leq (2C_1^n + C_1^{n-1} + \cdots + C_1^2)\mu(f).$$

B2i) If $A_\mu \neq (0)$, $\mu(f) < \infty$, then μ is a Krull no trivial (C_1, C_2)- Holder valuation of R.
B2ii) If $A_\mu \neq (0)$, $\mu(f) = \infty$, then μ is a trivial no Krull (C_1, C_2)- Holder valuation.

Proof. B1)If $A_\mu = (0)$, then for each $h \in R - \{0\}, \mu(h) = 0$, thus μ is trivial (C_1, C_2)- Holder valuation on R.
B2) If $A_\mu \neq (0)$, then $f \neq 0$ and for all $g \in R - \{0\}$, such that $\mu(g) \neq 0$, we obtain that $\mu(g) \neq 0$. Thus, $g \in A_\mu$ and there exists, $h \in R - A_\mu$ such that, $g = hf^n$. Hence

$$C_1^{-1}(0 + \mu(f^n)) = C_1^{-1}(\mu(h) + \mu(f^n)) \leq \mu(g)$$
$$= \mu(hf^n) \leq C_1(\mu(h) + \mu(f^n)) = C_1(0 + \mu(f^n)),$$

thus by lemma 2.2 we have

$$(2C_1^{-n} + C_1^{1-n} + \cdots + C_1^{-2})\mu(f) \leq \mu(g)$$
$$\leq (2C_1^n + C_1^{n-1} + \cdots + C_1^2)\mu(f).$$

B2i)Let $\mu(f) < \infty$, then by B2 for all $g \in R - \{0\}$, we have $\mu(g) < \infty$, $\mu(g) \neq 0$. Therefore μ is a Krull (C_1, C_2)- Holder valuation, but μ is not trivial.
B2ii) Let $\mu(f) = \infty$. Then by B2 for all $g \in R - \{0\}$, such that $\mu(g) \neq 0$, we have $\mu(g) = \infty$. Therefore μ is trivial, but μ is not Krull (C_1, C_2)- Holder valuation. □

Corollary 2.6. Let $\mu, \bar{\mu} \in Hval(R), \mu \preceq \bar{\mu}$. Then we have:
I) If μ is trivial (C_1, C_2)- Holder valuation on D, then $\bar{\mu}$ is trivial (C_1, C_2)- Holder valuation on D.
II) If μ is of type A, then $\bar{\mu}$ can be either of type A or B.
II) If μ is of type B1, then $\bar{\mu}$ can be either of type B1 or B2.
III) If μ is of type B2, then either $\bar{\mu}$ is of type B2i such that, $\mu(f) < \bar{\mu}(f) < \infty$ or $\bar{\mu}$ is of type B2ii such that, $\mu(f) \leq \bar{\mu}(f) = \infty$.

Proof. by remark 2.1 and definition it is clear. □

We next set some notation that we shall use throughout the paper and which is similar to some one of [5]. Let $\mu, \bar{\mu} \in HVal(R)$ be such that $\mu \preceq \bar{\mu}$. We denote by
$\bar{\Phi}(\mu, \bar{\mu}) = \{\varphi \in R; \mu(\varphi) < \bar{\mu}(\varphi)\}$: Note that,
$\bar{\Phi}(\mu, \bar{\mu}) = \{\varphi \in R; \mu(\varphi) < \bar{\mu}(\varphi)\} = \emptyset$ if and only if $\mu = \bar{\mu}$.
Furthermore, if $\bar{\Phi}(\mu, \bar{\mu}) \neq \emptyset$, we write:
1) $d(\mu, \bar{\mu}) = min\{deg\varphi; \varphi \in \bar{\Phi}(\mu, \bar{\mu})\}$.
2) $\Phi(\mu, \bar{\mu}) = \{\varphi \in \bar{\Phi}(\mu, \bar{\mu}); deg\varphi = d(\mu, \bar{\mu}) \quad and \quad \varphi \quad is \quad monic\}$.
3)$\Lambda(\mu, \bar{\mu}) = \{\bar{\mu}(\varphi); \varphi \in \Phi(\mu, \bar{\mu})\} = \bar{\mu}(\Phi(\mu, \bar{\mu}))$.
4)$\gamma(\mu, \bar{\mu}) = sup(\Lambda(\mu, \bar{\mu})) \in \bar{\mathbf{R}}$.

Remark 2.7. Note that if $\varphi \in \Phi(\mu, \bar{\mu})$, then φ is an irreducible left skew polynomial and if $\mu' \in HVal(R)$ with $\mu \preceq \bar{\mu} \preceq \mu'$, then $d(\mu, \bar{\mu}) \geq d(\mu, \mu')$ and $d(\mu, \mu') \leq d(\bar{\mu}, \mu')$.
Because if φ is not an irreducible left skew polynomial, then there exists $f, g \in R$ such that $\varphi = fg$, $0 < deg(f) < deg(\varphi)$),$0 < deg(g) < deg(\varphi)$), since $\varphi \in \Phi(\mu, \bar{\mu})$, hence $\mu(f) = \bar{\mu}(f)$ and $\mu(g) = \bar{\mu}(g)$.Thus $\mu(\varphi) = \bar{(\mu)}(\varphi)$, which is contradiction. We finish this section with the following technical result.

Theorem 2.8. Let $\mu, \bar{\mu}, \mu' \in HVal(R)$ be such that $\mu \prec \bar{\mu} \preceq \mu'$. Then the following statements

hold.
a) $\bar{\mu}(\varphi) > \mu(\varphi)$ for each $\varphi \in \Phi(\mu, \mu^{'})$, in particular $d(\mu, \bar{\mu}) = d(\mu, \mu^{'})$ and $\Phi(\mu, \bar{\mu}) = \Phi(\mu, \mu^{'})$
b)Every totally ordered subset $S \subset HVal(R)$ is bounded above.

Proof. a) let there exists $\varphi \in \Phi(\mu, \mu^{'})$, such that $\bar{\mu}(\varphi) = \mu(\varphi)$.Then
$\mu(\varphi) < \mu^{'}(\varphi), d(\mu, \mu^{'}) = deg\varphi$.
onside since $\mu \prec \bar{\mu}$, thus there exists $\varphi^{'} \in \Phi(\mu, \bar{\mu})$.
Hence by remark 2.4 we have $deg\varphi^{'} = d(\mu, \bar{\mu}) \geq d(\mu, \mu^{'}) = deg(\varphi)$.
Therefore $\varphi^{'} = q\varphi + r$ with $q, r \in R$ and $deg(r) < deg(\varphi)$.
We have $deg(q) < deg(\varphi^{'}) = d(\mu, \bar{\mu})$. Thus $\bar{\mu}(q) = \mu(q)$, since $\bar{\mu}(\varphi) = \mu(\varphi)$, so $\bar{\mu}(q\varphi) = \mu(q\varphi)$. onside
$deg(q\varphi) = deg(\varphi^{'}) = d(\mu, \bar{\mu})$, hence $\mu(q\varphi) < \bar{\mu}(q\varphi)$, which is contradiction.
by remark 2.7 we have $d(\mu, \bar{\mu}) \geq d(\mu, \mu^{'})$, one sided let there exists $\varphi \in \phi(\mu, \mu')$, such that $d(\mu, \mu') = deg(\varphi)$, thus by assumption we have $\bar{\mu}(\varphi) > \mu(\varphi)$, so $\varphi \in \bar{\phi}(\mu, \bar{\mu})$, thus $d(\mu, \bar{\mu}) \leq deg(\varphi) = d(\mu, \mu')$. Therefore $d(\mu, \mu') = d(\mu, \bar{\mu})$, by definition ϕ it is clear that $\Phi(\mu, \bar{\mu}) = \Phi(\mu, \mu^{'})$.

b) let $\mu^* : R \rightarrow \bar{\mathbf{R}}$ be given by $\mu^*(f) = sup\{\mu_*(f); \mu_* \in S\}$. Since S is a totally ordered set, thus μ^* is well defined. We shall now show that $\mu^* \in HVal(R)$, and hence μ^* is an upper bound of S. We only need to statements (HV1) and (HV2) of Definition of(C_1, C_2)- Holder valuation for μ^*. Since $S \subset HVal(R)$, thus $C_1^{-1}(\mu_*(f) + \mu_*(g)) \leq \mu_*(fg) \leq C_1(\mu_*(f) + \mu_*(g))$ for all $\mu_* \in S$.Thus $C_1(\mu_*(f) + \mu_*(g))$ is upper bound for $\mu_*(fg)$, therefore $\mu^*(fg) \leq C_1(\mu_*(f) + \mu_*(g)) \leq C_1(\mu^*(f) + \mu^*(g))$. Onside let $\epsilon > 0$, therefore $\mu^*(f) - \epsilon/2, \mu^*(g) - \epsilon/2$ are not upper bound, thus there exist $\mu_1, \bar{\mu}_1 \in S$ such that, $\mu^*(f) - \epsilon/2 \leq \mu_1(f)$, $\mu^*(g) - \epsilon/2 \leq \bar{\mu}_1(g)$. Since S is totally ordered set, we can also assume without loss of generality$\mu_1 \preceq \bar{\mu}_1$. therefore $\mu^*(f) - \epsilon/2 \leq \bar{\mu}_1(f)$,

$$\mu^*(fg) \geq \bar{\mu}_1(fg) \geq C_1^{-1}(\bar{\mu}_1(f) + \bar{\mu}_1(g))$$

$$\geq C_1^{-1}(\mu^*(f) - \epsilon/2 + \mu^*(g) - \epsilon/2) = C_1^{-1}(\mu^*(f) + \mu^*(g)) - C_1^{-1}\epsilon/2$$

. since ϵ is arbitrary element, put $\epsilon = 1/n$. so,

$$\mu^*(fg) \geq C_1^{-1}(\mu^*(f) + \mu^*(g)) - C_1^{-1}\frac{\epsilon}{2n}.$$

since $\mu^*(fg), \mu^*(f), \mu^*(g) \in \mathbb{R}$ and $\mathbb{R}$ is metric space,thus

$$\lim_{n \longrightarrow \infty} \mu^*(fg) \geq \lim_{n \longrightarrow \infty} C_1^{-1}(\mu^*(f) + \mu^*(g)) - \lim_{n \longrightarrow \infty} {C_1}^{-1}\frac{1}{2n}.$$

Therefore

$$\mu^*(fg) \geq C_1^{-1}(\mu^*(f) + \mu^*(g)).$$

Also

$$\mu^*(f + g) \geq \bar{\mu}_1(f + g) \geq C_2 Min\{\bar{\mu}_1(f), \bar{\mu}_1(g\}$$

$$\geq C_2 Min\{\mu^*(f) - \epsilon/2, \mu^*(g) - \epsilon/2\}.$$

Let $\mu^*(f) \leq \mu^*(g)$.Then

$$\mu^*(f) - \epsilon/2 \leq \mu^*(g) - \epsilon/2.$$

Thus

$$\mu^*(f + g) \geq C_2(\mu^*(f) - \epsilon/2),$$

put $\epsilon = \dfrac{1}{n}$. hence

$$\lim_{n\longrightarrow\infty} \mu^*(f+g) \geq \lim_{n\longrightarrow\infty} C_2(\mu^*(f)) - \lim_{n\longrightarrow\infty} C_2/2n.$$

Thus

$$\mu^*(f+g) \geq C_2\mu^*(f)) = C_2 Min\{\mu^*(f), \mu^*(g)\}.$$

Therefore

$$\mu^*(f+g) \geq C_2 Min\{\mu^*(f), \mu^*(g)\}.$$

3) For each $\mu_* \in S$, we have $\mu^*(0) \geq \mu_*(0) = \infty$, thus $\mu^*(0) = \infty$. Therefore $\mu^* \in Hval(R)$. □

3 Augmented and Limit Valuations and MacLane Key Polynomials

We begin by introducing some notation. For each $g \in R$ we denote by $q(\varphi, g), r(\varphi, g)$ the unique elements of R such that $\varphi.g = q(\varphi, g)\varphi + r(\varphi, g)$ with $deg(r(\varphi, g)) < deg(\varphi)$,and $degq(\varphi, g) = deg(g)$, i.e. the left quotient and the left rest in the left division of $\varphi.g$ by φ. Throughout this section, $\mu, \bar{\mu} \in HVal(R)$ will be two fixed real Holder valuations such that $\mu \prec \bar{\mu}$, Since $\Phi(\mu, \bar{\mu}) \neq \emptyset$, we also fix $\varphi \in \Phi(\mu, \bar{\mu})$. Next technical result relates the properties of the left division by φ with the order $\preceq$.

Lemma 3.1. With the above assumptions and notation, let $g, f \in R$ be such that $0 \leq deg(g) < deg(\varphi) < deg(f)$. The following statements hold.

(i) $\bar{\mu}(g) = \mu(g) = \bar{\mu}(q(\varphi, g)) = \bar{\mu}(q(\varphi, g)) < C_1\mu(r(\varphi, g)) - \bar{\mu}(\varphi)$
(ii) Let $\varphi^n.g = g_n^{(n)}\varphi^n + g_{n-1}^{(n)}\varphi^{n-1} + \cdots + g_0^{(n)}$,such that $deg(g_i^i) < deg(\varphi)$, $0 \leq i \leq n-1$ and $deg(g_n^{(n)}) = deg(g)$. Then
$C_1^{-2}\bar{\mu}(g_n^{(n)}\varphi^n) \leq \bar{\mu}(\varphi^n.g) \leq C_1^2\bar{\mu}(g_n^{(n)}\varphi^n) \leq C_1^2\bar{\mu}(g_i^{(i)}\varphi^i)$ for $0 \leq i \leq n-1$.

Proof. (i)We have $deg(g) < deg(\varphi)$ and $deg(r(\varphi, g)) < deg(\varphi)$ and

$deg(q(\varphi, g)) = deg(g)$. Thus $\bar{\mu}(r(\varphi, g)) = \mu(r(\varphi, g))$ and $\bar{\mu}(q(\varphi, g)) = \mu(q(\varphi, g)) = \mu(g) = \bar{\mu}(g)$.
Suppose that $\bar{\mu}(q(\varphi, g)) \geq C_1\mu(r(\varphi, g)) - \bar{\mu}(\varphi)$.
Hence, $\bar{\mu}(q(\varphi, g).\varphi) \geq C_1^{-1}(\bar{\mu}(q(\varphi, g)) + \bar{\mu}(\varphi)) \geq C_1^{-1}C_1\mu(r(\varphi, g)) = \bar{\mu}(r(\varphi, g))$
and $\bar{\mu}(r(\varphi, g)) \geq C_2 min\{\bar{\mu}(q(\varphi, g).\varphi), \bar{\mu}(\varphi.g)\} = C_2\bar{\mu}(q(\varphi, g).\varphi)$.
So, $\bar{\mu}(q(\varphi, g).\varphi) \geq C_2\bar{\mu}(q(\varphi, g).\varphi)$, which is a contradiction.

The result follows.

(ii) Since $C_1^{-2}\bar{\mu}(fg) \leq \bar{\mu}(gf) \leq C_1^2\bar{\mu}(fg)$ for each $\bar{\mu} \in HVal(R)$,then the result easily follows from (i). □

Proposition 3.2. We assume all assumptions and notation 0f lemma 3.1 and let $I(\sigma, \delta, \mu, \bar{\mu}, \varphi) = min\{C_1\mu(r(\varphi, g) - \mu(g); g \in R, 0 \leq deg(g) < deg(\varphi)\}$.Then $I(\sigma, \delta, \mu, \bar{\mu}, \varphi) \in \bar{\mathbf{R}}$ and $I(\sigma, \delta, \mu, \bar{\mu}, \varphi) \geq \bar{\mu}(\varphi) > \mu(\varphi)$.

Proof. By lemma 3.1(i) we have that $C_1\mu(r(\varphi, g)) - \bar{\mu}(\varphi) \geq \mu(g)$ with $0 \leq deg(g) < deg(\varphi)$. Thus, $C_1\mu(r(\varphi, g)) - \mu(g) \geq \bar{\mu}(\varphi) \geq \mu(\varphi)$, for all $g \in R$, with $0 \leq deg(g) < deg(\varphi)$.so, $I(\sigma, \delta, \mu, \bar{\mu}, \varphi) \in \bar{\mathbf{R}}$ and
$I(\sigma, \delta, \mu, \bar{\mu}, \varphi) \geq \bar{\mu}(\varphi) > \mu(\varphi)$ □

In this section, we shall define left key skew polynomials for Krull (C_1, C_2)- Holder valuations in a similar way as in [5]. In fact, our concept of left key skew polynomial coincides with MacLane's one [5] when we only consider the polynomial ring in one variable with coefficients in a commutative field, (i.e. when D is a commutative field, $\sigma = 1_D$ and $\delta = 0$ [6].

With the notation as in the previous sections, let $\mu \in Hval(R)$ be a Krull (C_1, C_2)- Holder real valuation.

Definition 3.3. For any $f, g \in R$ we say f is μ-equivalent to g, if $\mu(f - g) > \mu(f) = \mu(g)$ and We shall denote it by $f \sim_\mu g$ or simply by $f \sim g$ when no confusion can arise.Moreover we say that g is left μ-divisible by f, if there exists $h \in R$ such that $g \sim_\mu hf$.

Definition 3.4. A non-zero element $\varphi \in R$ is a left key skew polynomial for μ, if it satisfies the following conditions:
(K.1) Irreducibility. Let $f, g \in R$ be such that fg is left μ-divisible by φ, then one of the factors is left μ-divisible by φ.
(K.2) Minimal degree. For all $f \in R$ such that f is left μ-divisible by φ, we have $deg(\varphi) \leq deg(f)$.
(K.3) Monicity. The leading coefficient of φ is 1.
(K.4) Compatibility.
$\mu(\varphi) < min\{C_1\mu(r(\varphi, g)) - \mu(g)\ ;\ g \in R;\ 0 \leq deg(g) < deg(\varphi)\}$
where $\varphi.g = q(\varphi, g)\varphi + r(\varphi, g)$ with $deg(r(\varphi, g)) < deg(\varphi)$,and $degq(\varphi, g) = deg(g)$
For a left key skew polynomial $\varphi \in R$, we write
$I(\sigma, \delta, \mu, \bar{\mu}, \varphi) = min\{C_1\mu(r(\varphi, g) - \mu(g); g \in R, 0 \leq deg(g) < deg(\varphi)\}$
and we call $I(\sigma, \delta, \mu, \bar{\mu}, \varphi)$ the left compatibility index of φ with respect to μ. Thus, the compatibility property means

$$I(\sigma, \delta, \mu, \bar{\mu}, \varphi) > \mu(\varphi).$$

In a similar way as in proposition 3.2, we have the following result.

Proposition 3.5. We consider all the assumptions and notation mentioned above and let φ be a left key skew polynomial for μ and $\tau \in \bar{R}$ be such that $I(\sigma, \delta, \mu, \bar{\mu}, \varphi) \geq \tau > \mu(\varphi)$, $\mu_\tau(g) = min\{C_1(\mu(g_i) + i\tau); 0 \leq i \leq r\}$ for each $g \in R$, where $g = \sum_{i=o}^{r} g_i\varphi^i$ with $deg(g_i) < deg(\varphi)$, $0 \leq i \leq r$. Then $\mu_\tau \in HVal(R)$. Furthermore, $\mu \leq \mu_\tau$ and $\mu_\tau(f) = C_1\mu(f)$ for each $f \in R$ such that $deg(f) < deg(\varphi)$.

Proof. Note that $\mu_\tau(0) = C_1\mu(0) = \infty$ and we have that Hv(1) is satisfied.Next, we show that (Hv(2),Hv(3)) are satisfied. in fact, let $f, g \in R$ such that $f = \sum_{i=o}^{r} f_i\varphi^i$,$g = \sum_{i=o}^{r} g_i\varphi^i$
with $deg(f_i) < deg(\varphi)$, $deg(g_i) < deg(\varphi)$, $0 \leq i \leq r$.

Thus, $f + g = \sum_{i=o}^{r}(f_i + g_i)\varphi^i$ and we have $\mu_\tau(f + g) = C_1(\mu(f_i + g_i) + i\tau)$ for some i consequently,
$\mu_\tau(f + g) \geq C_1C_2min\{\mu(f_i) + i\tau), \mu(g_i) + i\tau)\} =$
$C_2min\{C_1(\mu(f_i) + i\tau), C_1(\mu(g_i) + i\tau)\} \geq C_2min\{\mu_\tau(f), \mu_\tau(g)\}$and also

$C_1^{-1}(\mu_\tau(f) + \mu_\tau(g)) \leq \mu_\tau(fg) \leq C_1(\mu_\tau(f) + \mu_\tau(g))$.Hence, $\mu_\tau \in HVal(R)$.

For each $f \in R$ such that $deg(f) < deg(\varphi)$,we have that $f = f$ and it follows that $\mu_\tau(f) = C_1(\mu(f) + 0\tau) = C_1\mu(f)$.

For each $g \in R$ there exists $i \in \{0, 1, \cdots, r\}$ such that

$$\mu_\tau(g) = C_1(\mu(g_i) + i\tau) \geq C_1(\mu(g_i) + i\mu(\varphi)) \geq C_1(\mu(g_i) + \mu(\varphi^i))$$
$$\geq C_1 C_1^{-1}\mu(g_i\varphi^i) = \mu(g_i\varphi^i) \geq \mu(g).$$

□

Proposition 3.6. With the above assumptions and notation, let $\varphi \in R$ be a monic left skew polynomial. Then φ is a left key skew polynomial for μ if and only if there exists $\bar{\mu} \in HVal(R)$ such that $\mu \prec \bar{\mu}$ and $\varphi \in \phi(\mu, \bar{\mu})$.

Proof. The necessary condition is consequence of Proposition 3.5.

Conversely, suppose that there exists $\bar{\mu} \in HVal(R)$ such that $\mu \prec \bar{\mu}$ and $\varphi \in \phi(\mu, \bar{\mu})$.By the fact that monicity and compatibility properties with respect to μ are verified for every $\varphi \in \phi(\mu, \bar{\mu})$, we only need to prove the minimality degree and irreducibility properties with respect to μ that is, φ.In fact if $f \in R$ is left μ-divisible byφ and $deg(f) < deg(\varphi)$, then $\mu(f - h\varphi) > \mu(f) = \mu(h\varphi)$. Since, $\mu(f) = \bar{\mu}(f)$ and $\mu(h\varphi) < \bar{\mu}(h\varphi)$ and we obtain that $\bar{\mu}(f) = \mu(f)$ $< min\{\bar{\mu}(f - h\varphi), \bar{\mu}(h\varphi)\}$, on side $\bar{\mu}(f) \geq C_2 min\{\bar{\mu}(f - h\varphi), \bar{\mu}(h\varphi)\}$, which is a contradiction. In order to see the irreducibility property with respect to μ, let $f, g \in R$ be such that fg is left μ-divisible by φ and assume that neither f nor g are left μ-divisible by φ. Thus there exist $h \in R$ such that $\mu(fg - h\varphi) > \mu(fg) = \mu(h\varphi)$,and write $f = q_1\varphi + r(f)$ and $g = q_2\varphi + r(g)$ with $0 \leq deg(r(f)) < deg(\varphi), deg(r(g)) < deg(\varphi)$. By the fact that f is not left μ-divisible by φ, we have that $\mu(r(f)) \leq \mu(f)$. Moreover, if $\mu(r(f)) < \mu(f)$, then $\bar{\mu}(r(f)) = \mu(r(f)) < \mu(f) \leq \bar{\mu}(f)$ and $\bar{\mu}(r(f)) = \mu(r(f)) = \mu(q_1\varphi) < \bar{\mu}(q_1 f)$, which is a contradiction. Hence, $\mu(r(f)) = \mu(f)$ and by similar methods as above we obtain that $\mu(r(g)) = \mu(g)$. Note that $fg - h\varphi = k + r(f)r(g)$, where $k = q_1\varphi q_2\varphi + r(f)q_1\varphi + q_1\varphi r(g) - h\varphi$. Since $\mu(fg - h\varphi) > \mu(fg) \geq C_1^{-2}\mu(r(f)r(g)) \geq C_1^{-4}\bar{\mu}(r(f)r(g))$, then $\mu(k) \geq C_1^{-2}\mu(r(f)r(g)) \geq C_1^{-4}\bar{\mu}(r(f)r(g))$, and we have that $\bar{\mu}(fg - h\varphi) \geq \mu(fg - h\varphi) > \bar{\mu}(r(f)r(g))$ and $\bar{\mu}(k) > \mu(k) > C_1^{-4}\bar{\mu}(r(f)r(g))$, which is a contradiction. □

We finish this paper with the following example.

Example 3.7. Let $D = \mathbb{C}(X, \sigma)$ be the Ore quotient ring of $\mathbb{C}[X, \sigma, 0] = \mathbb{C}[X, \sigma]$, where σ is the conjugation automorphism on $\mathbb{C}$. Note that D is a division ring. Let δ be the inner derivation on D associated with $i \in \mathbb{C}$ (i.e. $\delta(a) = ia - ai$ for each $a \in D$.) Thus $\delta(X^{2n+1}) = 2iX^{2n+1}$, and $\delta(X^{2n}) = 0$. We write $R = D[T, 1_D, \delta] = D[T, \delta]$, let us also write degX the usual degree in $C[X, \sigma]$ and denote by ν the valuation $-degX$ on D. We have $\nu(\delta(P(X))) \geq \nu(P(X))$ for each $P(X) \in C[X; \sigma]$. In particular, $\nu(\delta(a)) \geq \nu(a)$ for each $a \in D$. Thus, we can consider $\mu_0 : R \longrightarrow \mathbf{R}$ the extension of ν given by $\mu_0(T) = 0$. (See [5], Proposition 4.5)

We note that $T - i$ is a central element of R, since δ is the inner derivation associated with i. By the fact that $T - i$ has degree one, it is easy to check that $T - i$ is a left skew key polynomial for μ_0 and obviously $I(1_D, \delta, \mu_0, T - i) = \infty$.

Competing Interests

The authors declare that no competing interests exist.

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